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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Gibbs Phenomenon and Lebesgue Constants", submitted by Fred Ustina, B.Sc., in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

Gibbs phenomenon concerns a special kind of non-uniform convergence of the partial sums of the Fourier series in the neighbourhood of a discontinuity of the function the series represents. Summability methods have been devised to make convergent series converge faster, and also to transform a divergent series into a convergent series in the region of definition of its associated function. These methods, being essentially averaging processes, lead to the question of their effectiveness in smoothing out the Gibbs phenomenon.

The divergence of a sequence of constants, now known as the Lebesgue constants, has been shown to imply the existence of a continuous function whose Fourier series diverges at a given point, and another such function whose series converges everywhere, but not uniformly in the neighbourhood of a prescribed point. The divergence of the transforms of the Lebesgue constants under a given summability method implies the existence of a continuous function whose Fourier series is not everywhere summable by the method, and of another such function whose Fourier series is everywhere summable, but not uniformly in the neighbourhood of a given point.

Gibbs phenomenon was discovered by H. Wilbraham (1848), and essentially rediscovered by J. Willard Gibbs (1899). P. du Bois-Reymond was the first to construct a continuous function whose Fourier series diverges at a point, and Henri Lebesgue (1906) proved that the divergence of the Lebesgue constants implied the existence of such a function. This thesis gives an account of the many investigations carried out in these topics.

CONTENTS

CHAPTER I

GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR ORDINARY CONVERGENCE

Gibbs phenomenon concerns the non-uniform convergence of the partial sums of the Fourier series at a point of discontinuity of the function the series represents. It was discovered by H. Wilbraham [122, 1848] who also worked out the mathematics of it for the case of a simple jump discontinuity. Little or no additional work appears to have been done on it until J. W. Gibbs [24, 1899] discovered it again, apparently unaware of Wilbraham's earlier result. Since then, many authors have contributed to its study and additional contributions are being made up to the present.

The phenomenon has been studied extensively under various summability methods. Its behaviour for these may then be used to characterize the methods themselves. The concept has been extended to multiple Fourier series, to discontinuities of the second kind, and to generalized Fourier series.

du Bois-Reymond* was the first to prove that there exists a continuous, periodic function whose Fourier series diverges at a given point. This phenomenon is now known as the du Bois-Reymond singularity. Lebesgue [59, 1906] proved that the divergence of a sequence of constants,

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* du Bois-Reymond, P., Untersuchungen über die Convergenz und Divergenz der Fourierschen Darstellungsformeln. Ab. Akad. d. Wissens. Munchen, 12 (1876), 1-103.

now known as the Lebesgue constants, implies the existence of such a function; and of another such function whose Fourier series converges everywhere, but not uniformly in the neighbourhood of a given point. This last phenomenon is known as the Lebesgue singularity.

The Lebesgue constants, L_n, for ordinary convergence are defined by the integral

$$L_n = \frac{2}{\pi} \int_0^{\pi} |D_n(t)|dt$$

where

$$D_n(t) = \frac{\sin(n + 1/2)t}{2 \sin 1/2\, t}$$

If the kernel $D_n(t)$ is replaced in the integral by its transform under a given summability method, the divergence of the corresponding constants, called the Lebesgue constants for that particular method, implies the existence of a continuous function whose Fourier series is not summable by that method, and of another such function whose Fourier series is everywhere summable, but not uniformly in the neighbourhood of a given point.

The Lebesgue constants have also been studied for various summability methods. The concept has been extended to Fourier series in orthonormal systems other than the trigonometric sustem, to multiple Fourier series, and to transformations in general Banach spaces.*

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* For an extension of the concept to transformations in general Banach spaces, see Goes [25].

In order to preserve a measure of consistency in what follows, the notation of the original authors has often been changed. Except in cases where such practice would lead to some confusion, for instance, equation (2.18), the letter ν has been used exclusively as a summation variable. The limits of summation have not been indicated whenever the summation extends from zero to infinity, or, in cases where the first term is meaningless or redundant, from one to infinity. As is customary, the symbol $\sim$ is used to denote the Fourier series representation of a function.

1. Gibbs phenomenon

Consider the function

$$(1.1)\qquad f(x) = \begin{cases} -\pi/2, & (2k-1)\pi < x < 2k\pi \\ \pi/2, & 2k\pi < x < (2k+1)\pi \\ 0\,, & x = k\pi, \end{cases}$$

where $k = 0, \pm 1, \pm 2, \ldots$. This function has a Fourier series expansion in the trigonometric system:

$$(1.2)\qquad f(x) \sim 2 \sum \frac{\sin(2\nu - 1)x}{2\nu - 1} \quad .$$

Denote the partial sums of this series by

$$(1.3)\qquad s_n(x) = 2 \sum_1^n \frac{\sin(2\nu - 1)x}{2\nu - 1} \quad .$$

Then $f(x)$ and $s_n(x)$ are odd, periodic functions of period 2π. Since $\sin(2\nu - 1)(\pi/2 + x) = \sin(2\nu - 1)(\pi/2 - x)$, both these functions are symmetric about $x = \pi/2$. Hence to determine their behaviour completely, it is sufficient to determine it for the interval $0 \leq x \leq \pi/2$.

Differentiating (1.3), we have

$$(1.4) \qquad s_n'(x) = 2 \sum_1^n \cos(2\nu - 1)x$$

$$= \frac{\sin 2nx}{\sin x} .$$

It follows that in the interval $(0, \pi/2]$ the zeros of $s_n'(x)$ are the zeros of $\sin 2nx$. Denoting the k^{th} zero by x_k, we have

$$(1.5) \qquad x_k = \frac{k\pi}{2n}, \qquad k = 1, 2, \ldots, n.$$

From the continuity of $s_n(x)$ and $s_n'(x)$, it is seen that odd values of k yield points of relative maxima of $s_n(x)$, and even values yield points of relative minima. At these points, $s_n(x)$ is given by

$$(1.6) \qquad s_n(x_k) = \int_0^{x_k} \frac{\sin 2nt}{\sin t}\, dt.$$

Since the integration in (1.6) cannot be performed by any known means, we obtain an asymptotic estimate.

$$(1.7) \qquad s_n(x_k) = \int_0^{x_k} \frac{\sin 2nt}{\sin t}\, dt$$

$$= \int_0^{x_k} \left\{\frac{t}{\sin t} - 1\right\} \frac{\sin 2nt}{t}\, dt + \int_0^{x_k} \frac{\sin 2nt}{t}\, dt.$$

The absolute value of the first integral on the right hand side does not exceed

$$(1.8) \qquad 2nx_k . \max \{\frac{t}{\sin t} - 1\} = k\pi \max \{\frac{t}{\sin t} - 1\}$$

$$= k\pi \max \{\frac{k\pi}{2n} . (\sin \frac{k\pi}{2n})^{-1} - 1\}$$

by (1.5), and by considering that $\frac{t}{\sin t}$ is an increasing function of t for $0 \leq t \leq \pi$. If we keep k bounded and let n tend to infinity, it is seen that this quantity can be made arbitrarily small. Setting $\lim_{n \to \infty} s_n(x_k) = s(x_k)$, we have

$$(1.9) \qquad s(x_k) = \int_0^{k\pi} \frac{\sin t}{t} dt$$

$$= \int_0^{\pi} \frac{\sin t}{t} dt + \int_{\pi}^{2\pi} \frac{\sin t}{t} dt + \ldots + \int_{(k-1)\pi}^{k\pi} \frac{\sin t}{t} dt.$$

Denoting the difference $s(x) - f(x)$ by $r(x)$ we have, for $k = 1, 2, \ldots, 6$*,

$$\begin{aligned} s(x_1) &= 1.851936 & r(x_1) &= 0.281140 \\ s(x_2) &= 1.418158 & r(x_2) &= -0.152638 \\ s(x_3) &= 1.674760 & r(x_3) &= 0.103964 \\ s(x_4) &= 1.492164 & r(x_4) &= -0.078632 \\ s(x_5) &= 1.633963 & r(x_5) &= 0.063167 \\ s(x_6) &= 1.518036 & r(x_6) &= -0.052760 \end{aligned}$$

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* Corey, S. A., The evaluation of $\int_0^{\pi} \frac{\sin mx}{x} dx$. Amer. Math. Monthly, 13 (1906), 12-13.

From this it is seen that as n tends to infinity, the first finite number of maxima and minima of $s(x)$ do not tend to $f(x)$, but rather tend to definite limits distinct from $f(x)$. This phenomenon is known as the Gibbs phenomenon.

The above development was motivated largely by Carslaw [8, 1921]. Reference was also made to Gronwall [27], Runge [90], Knopp [53], and Bocher [3].

So far, we concerned ourselves with the Fourier series of a specific function. It is of interest to note, however, that the partial sums of the Fourier series of any function, which is of bounded variation and has no removable discontinuities, shows the Gibbs phenomenon at every point of discontinuity and only there. For a proof of this result, the reader is referred to Zygmund [127, p. 61]. From the manner of proof, it follows that the behaviour of the Gibbs phenomenon for any such function is indicated by a study of the Gibbs phenomenon for a specific such function. The functions most commonly used for this study are $f(x) \sim \sum \frac{\sin \nu x}{\nu}$ and $f(x) \sim 2 \sum \frac{\sin (2\nu - 1)x}{2\nu - 1}$.

We now formulate some general definitions necessary in the study of the phenomenon.

Definition 1.1

Let $\{\epsilon_n\}$ be a sequence of positive numbers such that $\lim_{n \to \infty} \epsilon_n = 0$, and let $\{p_n\}$ be a sequence of natural numbers such that

$\lim_{n \to \infty} p_n = \infty$. Let $\{s_n(x)\}$ be a sequence of real valued functions defined for $0 < |x - x_o| < \epsilon_n$ for each n. Then

$$\limsup_{\substack{n \to \infty \\ x \to x_o}} s_n(x) = \limsup_{n \to \infty} \{s_m(x) : m > p_n, 0 < |x - x_o| < \epsilon_n\}$$

$$\liminf_{\substack{n \to \infty \\ x \to x_o}} s_n(x) = \liminf_{n \to \infty} \{s_m(x) : m > p_n, 0 < |x - x_o| < \epsilon_n\}.$$

<u>Definition 1.2</u>

Let $\{s_n(x)\}$ be a sequence of real valued functions converging to a limit function $f(x)$ for $0 < |x - x_o| < \epsilon$. The sequence $\{s_n(x)\}$ is said to exhibit the Gibbs phenomenon at $x = x_o$ if either or both of the following hold:

$$\limsup_{\substack{x \to x_o \\ n \to \infty}} \{s_n(x)\} > \limsup_{x \to x_o} f(x)$$

$$\liminf_{\substack{x \to x_o \\ n \to \infty}} \{s_n(x)\} < \liminf_{x \to x_o} f(x) .$$

<u>Definition 1.3</u>

Let $\{s_n(x)\}$ be defined as in Definition 1.2. The Gibbs set, GS , of the sequence $\{s_n(x)\}$ at a point x_o , is the union of all numbers $c = \lim_{n \to \infty} s_n(x_n)$ as x_n tends to x_o through appropriate sequences.

Definition 1.4

Let $\{s_n(x)\}$ again be defined as in Definition 1.2. Let $f(x)$ be of bounded variation in some interval containing x_o, a point of simple discontinuity of $f(x)$, in its interior. The Gibbs ratio, GR, is defined by

$$\left| \frac{\displaystyle\limsup_{\substack{n \to \infty \\ x \to x_o}} s_n(x) - \liminf_{\substack{n \to \infty \\ x \to x_o}} s_n(x)}{\displaystyle\limsup_{x \to x_o} f(x) - \liminf_{x \to x_o} f(x)} \right|$$

Definition 1.5

Let $\{s_n(x)\}$ be a sequence of functions converging to a limit function $f(x)$ for $0 < |x - x_o| < \epsilon$, and let $f(x)$ have a pole at x_o. Set

$$s^+(x_o) = \limsup_{x_n \to x_o} \frac{s_n(x_n)}{|f(x_n)|}$$

$$s^-(x_o) = \liminf_{\substack{x_n \to x_o \\ n \to \infty}} \frac{s_n(x_n)}{|f(x_n)|}$$

for appropriate sequences $\{x_n\}$ such that $x_n \to x_o$ as $n \to \infty$. The Gibbs pseudo ratio, PR, is defined by

$$PR = |s^+(x_o) - s^-(x_o)| + 1$$

if the pole is of even order; and by

$$PR = (1/2)|s^+(x_o) - s^-(x_o)|$$

if the pole is of odd order.

Using the known result that if

$$\lim_{\substack{x \to x_o \\ n \to \infty}} g_n(x) = L, \qquad 0 < L < \infty ,$$

then

$$\limsup_{\substack{x \to x_o \\ n \to \infty}} g_n(x)h_n(x) = L \,.\, \limsup_{\substack{x \to x_o \\ n \to \infty}} h_n(x) ,$$

we show that the definition of the pseudo ratio for functions which have a pole of odd order is a generalization of the definition of the Gibbs ratio, GR , restricted to functions which are odd about their point of discontinuity. For then

$$\begin{aligned} PR &= 1/2\left|s^+(x_o) - s^-(x_o)\right| \\ &= 1/2\left|\limsup_{\substack{x_n \to x_o \\ n \to \infty}} \frac{s_n(x_n)}{|f(x_n)|} - \liminf_{\substack{x_n \to x_o \\ n \to \infty}} \frac{s_n(x_n)}{|f(x_n)|}\right| \\ &= \left|\limsup_{\substack{x_n \to x_o \\ n \to \infty}} \frac{s_n(x_n)}{2|f(x_n)|} - \liminf_{\substack{x_n \to x_o \\ n \to \infty}} \frac{s_n(x_n)}{2|f(x_n)|}\right| \\ &= \left|\frac{\limsup\limits_{\substack{x_n \to x_o \\ n \to \infty}} s_n(x_n) - \liminf\limits_{\substack{x_n \to x_o \\ n \to \infty}} s_n(x_n)}{\limsup\limits_{x_n \to x_o} f(x_n) - \liminf\limits_{x_n \to x_o} f(x_n)}\right| \\ &= GR , \end{aligned}$$

by comparison with Definition 1.4. The last result follows by observing that

$$\begin{aligned} \lim_{x_n \to x_o} 2|f(x_n)| &= 2 \limsup_{x_n \to x_o} f(x_n) = -2 \liminf_{x_n \to x_o} f(x_n) \\ &= \limsup_{x_n \to x_o} f(x_n) - \liminf_{x_n \to x_o} f(x_n) . \end{aligned}$$

There is no similar analogue to the corresponding definition of the pseudo ratio for functions which have a pole of even order.

For similar definitions of the Gibbs phenomenon, Gibbs ratio and Gibbs set, the reader is referred to Forbes [22], where also is found a discussion of some of the definitions of Gibbs phenomenon appearing in some of the literature.

Now let $f(x)$ have a simple jump discontinuity at a point x_o. The jump is defined by

$$d = |f(x_o+) - f(x_o-)|.$$

Let

$$c = (1/2)\{f(x_o+) + f(x_o-)\}. \tag{1.11}$$

For a large class of functions, including the example treated in this section, the Gibbs set is a closed, finite interval of length greater than d, and is symmetric about c. That this is not always the case was proved by Hardy and Rogosinski [33, 1943]. They constructed the function

$$f(x) = g(x) + \sum \frac{\sin \nu x}{\nu} \tag{1.12}$$

where

$$\begin{aligned} g(x) &= h_k^{1/2} \sin n_k(x - h_k)\,\operatorname{sgn}(x - h_k), \quad x \in (1/2h_k, 3/2h_k) \\ &= 0 \text{ elsewhere} \end{aligned}$$

and

$$h_k = \pi/4^{2k}, \qquad n_k = 4^{4^{k+1}}, \qquad k = 1, 2, \ldots .$$

The function $f(x)$ is continuous except for a jump at $x = 0$, and has a Gibbs set which is finite but not symmetric about c.

2. The Lebesgue Constants

Let $f(x)$ be Lebesgue integrable and periodic of period 2π, and let

$$(2.1) \qquad a_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos \nu t \, dt; \quad b_\nu = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin \nu t \, dt$$

$$(2.2) \qquad f(x) \sim \frac{1}{2} a_o + \sum (a_\nu \cos \nu x + b_\nu \sin \nu x) \quad .$$

The partial sums of (2.2) are given by

$$\begin{aligned}(2.3) \quad s_n(f;x) &= \frac{1}{2} a_o + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x) \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} f(t)dt + \frac{1}{\pi}\sum_1^n \int_{-\pi}^{\pi} f(t) \cos \nu(t - x)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + t)\{ \frac{1}{2} + \sum_1^n \cos \nu t\}dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x + t)\, D_n(t)\, dt\end{aligned}$$

where

$$\begin{aligned}(2.4) \qquad D_n(t) &= 1/2 + \sum_1^n \cos \nu t \\ &= \frac{\sin (n + 1/2)t}{2 \sin 1/2\, t} \quad .\end{aligned}$$

From (2.3) we have

$$(2.5)\qquad |s_n(f;\, x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)\, D_n(t)\, dt\right|$$

$$\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x+t)|\, |D_n(t)|\, dt.$$

If we restrict the argument to the class of functions for which $|f(x)| \leq 1$, then

$$(2.6)\qquad |s_n(f;x)| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |D_n(t)|\, dt$$

$$= \frac{2}{\pi}\int_{0}^{\pi} |D_n(t)|\, dt.$$

The Lebesgue constants, L_n , are defined by this last integral,

$$(2.7)\qquad L_n = \frac{2}{\pi}\int_{0}^{\pi} |D_n(t)|\, dt .$$

To see that equality can actually be attained in (2.6), we set $x = 0$ and $f(t) = \operatorname{sgn} D_n(t)$ in (2.5).

The function $\operatorname{sgn} D_n(t)$ has a finite number of jump discontinuities in $[-\pi, \pi]$, and so for each n , there exists a discontinuous function, $\operatorname{sgn} D_n(t)$, for which $|s_n(f;x)| = L_n$ for some x . Now let $\{f_{n,\nu}(t)\}$ be a sequence of continuous functions approaching the function $\operatorname{sgn} D_n(t)$ in the L^1 norm, and such that $|f_{n,\nu}(t)| \leq 1$ for all ν, t. Then

$$(2.8)\quad |s_n(f_{n\nu};\,0\,) - L_n|$$

$$= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f_{n,\nu}(t)\, D_n(t)\, dt - \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn} D_n(t).D_n(t)\, dt\right|$$

$$\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f_{n,\nu}(t) - \operatorname{sgn} D_n(t)|\, .\, |D_n(t)|\, dt$$

$$< \frac{(n + 1/2)}{\pi}\int_{-\pi}^{\pi} |f_{n,\nu}(t) - \operatorname{sgn} D_n(t)|\, dt.$$

By the convergence of the sequence $\{f_{n,\nu}(t)\}$ to the function $\operatorname{sgn} D_n(t)$, it follows that for any fixed n, the last integral can be made smaller than any small positive number $\frac{\pi\epsilon}{n+1/2}$ by a suitable choice of ν. Thus for each n, there exists a continuous function the n^{th} partial sum of whose Fourier series does not differ from the Lebesgue constant L_n by more than any small preassigned quantity ϵ. By the principle of uniform boundedness, the divergence of the Lebesgue constants implies the existence of a continuous function whose Fourier series diverges at a point*.

The above development was motivated largely by Zygmund [127]. For a complex function - theoretic approach, the reader is referred to Watson [117]. Many authors have contributed to the asymptotic estimates of the Lebesgue constants, notably Fejér, Gronwall, Szegö and Lorch. Others have extended the concept to multiple Fourier series, and to

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* For an engaging discussion of this, see Goffman, C., and Pedrick, G., First Course in Functional Analysis. Prentice Hall, Inc., (1965), p. 76.

Fourier series in other orthonormal systems. Goes [25] defined the Lebesgue constants for transformations in general Bonach spaces. Some of these extensions will be discussed in later chapters.

Fejér [20, 1910] proved that

$$(2.9) \quad L_n = \frac{4}{\pi^2}\log n + \frac{4}{\pi^2}\int_0^{\pi/2}\left(\frac{1}{\sin t} - \frac{1}{t}\right)dt + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt + o(1); \quad n\to\infty.$$

Gronwall [28, 1912] showed that

$$(2.10) \quad \frac{4}{\pi^2}\int_0^{\pi/2}\left(\frac{1}{\sin t} - \frac{1}{t}\right)dt = \frac{4}{\pi^2}\log\frac{4}{\pi}$$

and obtained the form

$$(2.11) \quad L_n = \frac{4}{\pi^2}\log n + h_n, \quad \frac{4}{\pi^2} + \frac{9}{4} > h_n > \frac{4C}{\pi^2},$$

where C is the Euler-Mascheroni constant. Lorch [66, 1945] obtained the form

$$(2.12) \quad L_n = \frac{4}{\pi^2}\log n + \frac{4}{\pi^2}\log 4 + \frac{2}{\pi}\int_0^1 \frac{\sin t}{t}\,dt - \frac{2}{\pi}\int_1^\infty\left\{\frac{2}{\pi} - |\sin t|\right\}dt + o(1), \quad n\to\infty,$$

thereby eliminating the gamma function from the representation (2.9). He showed that the constant term in (2.12) equals 1.270353... , and that

$$\int_1^\infty\left\{\frac{2}{\pi} - |\sin t|\right\}dt = -0.1668407.... .$$

Lorch [69, 1954] also obtained a very simple proof of the result

$$(2.13) \quad L_n = \frac{4}{\pi^2}\log n + O(1).$$

Fejér (ibid) also obtained the representation

$$(2.14) \qquad L_n = \frac{1}{2n+1} + \frac{2}{\pi} \sum_1^n \frac{1}{\nu} \tan \frac{\pi\nu}{2n+1} .$$

We mention here two other independent derivations of this representation, one by Szegö [108, 1921], and another by Carlitz [6, 1961].

Szegö also proved the absolute monotonicity of the constants L_n, and proved that

$$(2.15) \qquad L_n = \frac{16}{\pi^2} \sum \left(1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2\nu(2n+1)-1}\right) \frac{1}{4\nu^2-1} .$$

Watson [117, 1930] obtained the expression

$$(2.16) \qquad L_n = \frac{4}{\pi^2} \left\{ \log(2n+1) + a_o + \sum \frac{(-1)^{\nu-1} a_\nu}{(2n+1)^{2\nu}} \right.$$

where

$$(2.17) \qquad a_o = 2 \sum \frac{\log \nu}{4\nu^2-1} - \frac{\Gamma'(1/2)}{\Gamma(1/2)} = 2.441\ldots ,$$

and

$$(2.18) \qquad a_\nu = \frac{2^{2\nu-1}-1}{\nu} B_\nu \left(1 - \sum_1^\nu \frac{B_k \pi^{2k}}{2k!}\right), \qquad \nu \geq 1,$$

where the B_i are the Bernoulli numbers. Hardy [31, 1942] obtained the formulae

$$(2.19) \qquad L_n = \frac{4}{\pi^2} \int_o^\infty \frac{\sinh(2n+1)t}{\sinh t} \log \coth(n+1/2)t \, dt$$

and

$$(2.20) \qquad L_n = \int_o^\infty \frac{\tanh(2n+1)t}{(t^2+\pi/4)\tanh t} \, dt$$

from which he deduced Szegö's formula (2.15) and Watson's formula (2.16) respectively.

Sokolov [95, 1954] transformed the Lebesgue constants into a form suitable for numerical calculation. For small n, he used Fejér's formula (2.14), and tabulated the constants for $n = 1$ to 10. For large n, he obtained the representation

$$L_n = 0.40526 \log(2n+1) + 0.98818 + \frac{0.0120}{(2n+1)^2} + R_n \tag{2.21}$$

where

$$R_n < \frac{0.1}{(2n+1)^3} .$$

CHAPTER II

GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR SUMMABILITY METHODS APPLIED TO ORDINARY FOURIER SERIES

In Chapter I we observed that as n increases, the first maxima of the n^{th} partial sums of the Fourier series of $2\sum \frac{\sin(2\nu - 1)t}{2\nu - 1}$ approach the limiting value $\int_0^{\pi} \frac{\sin \mu}{\mu}\, d\mu$. The abscissa of each of these maxima is given by $\frac{\pi}{2n}$ so that however close they may be, no two such maxima coincide. Since a summability method is essentially an averaging process, the question arises as to how effectively a given method will smooth out these maxima, or what amounts to the same thing, does the Gibbs phenomenon persist for a given summability method? The answer to this may then be used in the characterization of the various summability schemes.

The divergence of the Lebesgue constants for a given summability method, on the other hand, implies the existence of a continuous function whose transform, while converging everywhere to the function, does not do so uniformly in the neighbourhood of a given point (the Lebesgue singularity), and another such function whose transform fails to converge everywhere to the function (the du Bois-Reymond singularity).

It follows that the study of the behaviour of Gibbs phenomenon and Lebesgue constants for various summability methods is of some interest. We give here some results which have been obtained to date

for ordinary Fourier series. For convenience, $\psi(t)$ will be used to denote the series $\sum \nu^{-1} \sin \nu t$.

We also give, summary form, a description of the methods of summability used in the various investigations. For a fuller account of these methods, the reader is referred to standard references such as Hardy [32], Widder [121] and Zeller [126]. Other references are given in cases where a given method is not discussed adequately in the standard texts.

1. Abel (A, λ_n) means

For a given series $\sum a_\nu$ and a given sequence of numbers $0 \leq \lambda_o < \lambda_1 < \dots , \lambda_n \to \infty$, if the series $\sum a_\nu e^{-\lambda_\nu x}$ is convergent for all $x > 0$, and if

$$(1.1) \qquad \sigma(x) = \sum a_\nu e^{-\lambda_\nu x} \to s, \qquad x \to 0 ,$$

the series $\sum a_\nu$ is said to be summable Abel (A, λ_n) to the sum s. In particular, for $\lambda_n = n^\lambda$, we have the (A, n^λ) means:

$$(1.2) \qquad \sigma(x) = \sum a_\nu e^{-\nu^\lambda x} .$$

The (A, n^λ) means are a particular case of the k'-methods of summability, obtained by taking $f(x) = \exp(-x^\lambda)$. The (A, λ_n) method is regular.

Kuttner [56, 1944] proved that for the Abel (A, n^{λ}) means of $\psi(t)$ not to exhibit the Gibbs phenomenon, it is necessary and sufficient that

$$(1.3) \qquad \Phi(z) = \int_0^{\infty} \exp(-x^{\lambda}) \frac{\sin zx}{x}\, dx \leq \pi/2$$

for all $z > 0$. He proved that $\Phi(z) - \pi/2$ takes on positive and negative values if $\lambda > 2$; and hence the Gibbs phenomenon persists for the (A, n^{λ}) means whenever $\lambda > 2$. Szász [107, 1952] proved that for this method, there is no Gibbs phenomenon when $0 < \lambda \leq 2$.

2. Barlaz $B_n(x_n)$ means

Let $s_k(t)$ denote the k^{th} partial sum of the series $\sum a_n(t)$. The Barlaz $B_n(x_n)$ means [2] are defined by the transform

$$(2.1) \qquad B_n(x_n) = e^{-x_n} \sum_{o}^{n} s_{\nu}(t) \frac{x_n^{\nu}}{\nu!} .$$

If $\lim B_n(x_n)$ exists as $x_n \to \infty$, $n \to \infty$, the sequence $\{s_n\}$ is said to be summable $B_n(x_n)$. A necessary and sufficient condition for the transform to be regular is that

$$(2.2) \qquad \lim (x_n - n)n^{-1/2} \to -\infty, \quad \text{as } x_n \to \infty, \ n \to \infty .$$

The Barlaz means are a variant of the Borel exponential means, obtained by replacing the continuous parameter x by the discrete variable x_n.

Forbes [22, 1962] proved that for the $B_n(x_n)$ means of the partial sums of $\psi(t)$, at $t = t_n$,

$$(2.3) \qquad \lim B_n(x_n) = \int_0^{x_n t_n} \frac{\sin \mu}{\mu}\, d\mu, \quad n \to \infty, \quad x_n \to \infty,$$
$$x_n t_n^3 < M < \infty^*.$$

Taking the sequence $\{t_n\}$ such that $x_n t_n = \pi$, it follows that the transform completely preserves the Gibbs phenomenon at $t = 0$.

3. Bernstein - Rogosinski $B_n^k(t)$ means

The n^{th} Bernstein-Rogosinski mean of order k $(k = 1, 2, \ldots)$ of the series $\sum a_\nu(t)$ is defined by the transform

$$(3.1) \qquad B_n^{(k)}(t) = \sum \cos^k \frac{\nu\pi}{2n+1}\, a_\nu(t) .$$

If as $n \to \infty$, $\lim B_n^{(k)}(t)$ exists, then the series $a_n(x)$ is said to be summable by the Bernstein-Rogosinski means of order k.

Harsiladze [35, 1955] proved that the $B_n^k(t)$ transform applied to $\psi(t)$ shows the Gibbs phenomenon at $t = 0$ for each $k = 1, 2, \ldots$, but in rapidly decreasing order since $d_k - \pi/2 < \{2^k(k+1)\}^{-1}$, where d_k denotes the length of the oscillation of the $B_n^k(t)$ means on the y axis. This result is of

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* The last condition, as originally stated by Forbes, was $x_n t_n^3 < \infty$.

some interest since the Bernstein-Rogosinski method is more powerful than the (C, r) method, and the (C, r) means of $\psi(t)$ do not show the Gibbs phenomenon for $r \geq r_o$, where r_o is Cramer's constant.

4. Borel $B_x(t)$ means

Let $\{s_n(t)\}$ be the sequence of partial sums of the series $\sum a_n(t)$. The Borel exponential means are defined by the transform

$$(4.1) \qquad B_x(t) = e^{-x} \sum s_\nu(t) \frac{x^\nu}{\nu!} .$$

If $\lim B_x(t)$, $x \to \infty$, exists, the sequence $\{s_n(t)\}$ is said to be summable by Borel's exponential means.

Borel integral means are defined by the transform

$$(4.2) \qquad B_x^*(t) = \int_o^\infty e^{-x} \sum a_\nu(t) \frac{x^\nu}{\nu!} \, dx.$$

The Borel exponential and integral means are regular. They are related by the identity

$$(4.3) \qquad B_x(t) = e^{-x} \sum a_\nu(t) \frac{x^\nu}{\nu!} + B_x^*(t) ,$$

and equivalent if and only if

$$(4.4) \qquad e^{-x} \sum a_\nu(t) \frac{x^\nu}{\nu!} \to 0 , \qquad x \to \infty .$$

The question of Gibbs phenomenon for these means was resolved by Lorch [70, 1957] who proved the following theorem:

Let $B_x(t)$ denote the x^{th} Borel exponential or integral mean of $\psi(t)$. Then, for a given τ, $0 \leq \tau \leq \infty$,

$$(4.5) \qquad \lim B_x(t_x) = \int_0^{\tau} \frac{\sin \mu}{\mu} \, d\mu \qquad x \to \infty$$

where $t_x \to 0+$ and $xt_x \to \tau$.

The theorem was first proved for the exponential means. To show that it is also true for the integral means, the author used the identity

$$(4.6) \qquad B_x(t) = e^{-x} \sum a_\nu \frac{x^\nu}{\nu!} + B_x^*(t)$$

where $B_x^*(t)$ is the integral mean and $a_\nu = \nu^{-1} \sin \nu t$.

From the theorem it follows that the Borel exponential and integral means show the same Gibbs phenomenon and have the same Gibbs set as the partial sums of $\psi(t)$.

The Lebesgue constants, $L_n(B)$, for Borel means were first examined by Moore [81, 1925] who pointed out that they become infinite and are of the same order as the Lebesgue constants for ordinary convergence. Lorch [65, 1944] replaced n by the continuous parameter x and obtained the asymptotic formula

$$(4.7) \qquad L_x(B) = \frac{2}{\pi^2} \log x + a_0 + O(x^{-1/2}) \qquad x \to \infty$$

where

$$(4.8) \qquad a_o = -\frac{2c}{\pi^2} - \frac{2}{\pi^2}\log\frac{\pi^2}{2} + 2\int_0^1 \log\Gamma(t)\cos\pi t\,dt$$

and C is the Euler-Mascheroni constant. To obtain this result, he effected a change of variable to get the transform of the constants into the form

$$(4.9) \qquad L_x(B) = \frac{2}{\pi}\int_0^{\pi/2} \exp(-2x\sin^2 t)\,\frac{|\sin(x\sin 2t+t)|}{\sin t}\,dt.$$

Then, if

$$(4.10) \qquad L_x(B_0) = \frac{2}{\pi}\int_0^{\pi/2} \frac{|\sin(2x+1)t|}{t}\,dt,$$

$$(4.11) \qquad L_x(B_1) = \frac{2}{\pi}\int_0^{\pi/2} \exp(-2x\sin^2 t)\,\frac{|\sin(x\sin 2t+t)|}{t}\,dt,$$

$$(4.12) \qquad L_x(B_2) = \frac{2}{\pi}\int_0^{\pi/2} \exp(-2xt^2)\,\frac{|\sin(x\sin 2t+t)|}{t}\,dt,$$

$$(4.13) \qquad L_x(B_3) = \frac{2}{\pi}\int_0^{\pi/2} \exp(-2xt^2)\,\frac{|\sin(2x+1)t|}{t}\,dt,$$

$$(4.14) \qquad d(x) = L_x(B_o) - L_x(B_3),$$

he proved that

$$\begin{aligned}(4.15) \qquad L_x(B) &= L_x(B_1) + O(x^{-1})\\ &= L_x(B_2) + O(x^{-1})\\ &= L_x(B_3) + O(x^{-1/2})\end{aligned}$$

and

$$(4.16) \qquad d(x) = \frac{2}{\pi^2} \log x + \frac{2C}{\pi^2} + \frac{2}{\pi^2} \log \frac{\pi^2}{2} + O(x^{-1/2}) \qquad x \to \infty .$$

Evaluating (4.10) and transforming the result, he obtained (4.7).

In a subsequent paper, Lorch [56, 1945] showed that

$$(4.17) \qquad a_o = -\frac{2C}{\pi^2} + \frac{2}{\pi^2} \log 2 + \frac{2}{\pi}\int_o^1 \frac{\sin \mu}{\mu} d\mu - \frac{2}{\pi}\int_1^\infty \{\frac{2}{\pi} - |\sin \mu|\}\frac{d\mu}{\mu} ,$$

and obtained its numerical value:

$$a_o = 0.732002\ldots .$$

5. Cesáro (C, r) means

For a given series $\sum a_\nu$ and integer $r \geq 0$, let

$$(5.1) \qquad A_n^{(0)} = a_o + a_1 + \ldots + a_n$$

$$(5.2) \qquad A_n^{(r)} = A_o^{(r-1)} + A_1^{(r-1)} + \ldots + A_n^{(r-1)}$$

$$(5.3) \qquad E_n^{(r)} = A_n^{(r)} \text{ for } a_o = 1 \text{ and } a_n = 0 \text{ for } n \geq 1$$

$$(5.4) \qquad C_n^{(r)} = \{A_n^{(r)}\}/\{E_n^{(r)}\} .$$

If $\lim C_n^{(r)}$ $(n \to \infty)$ exists and equals s, then the series $\sum a_n$ is said to be summable (C, r) to sum s .

It can be shown that $A_n^{(r)} = \sum \binom{\nu + r}{r} a_{n-\nu}$; $E_n^{(r)} = \binom{n + r}{r}$. Hence $C_n^{(r)} = \sum \binom{\nu + r}{r} \binom{n + r}{r}^{-1} a_{n-\nu} = \sum \frac{(\nu+r)!}{(n+r)!} \frac{n!}{\nu!} a_{n-\nu}$.

The Cesáro means of order r are also easily obtained from the regular Hausdorff means by taking $\mu_n = \binom{n+r}{r}^{-1}$ for the elements of the transformation matrix (μ_n), or by taking $\alpha(x) = 1 - (1-x)^r$ for the generating function of the constants μ_n. The extension to non-integral order r is then obvious.

The first study of Gibbs phenomenon for Cesáro means was carried out by Cramér [18, 1919], who showed that there exists a constant r_o, $0 < r_o < 1$, such that the Gibbs phenomenon will be present for the (C, r) means of $\psi(t)$ for $r < r_o$, but not for $r \geq r_o$. On the basis of some work by Lyons, Carslaw [9, 1926] conjectured that this constant was near 0.4. Gronwall [30, 1930] proved r_o to equal 0.4395516, where the last figure is in doubt by one unit. Gronwall obtained this result possibly as early as 1925, prior to Carslaw's conjecture, but he did not publish it until 1930 [30, p. 233, footnote].

Independently of the work of Gronwall, Cooke [16, 1930] proved that $0.40 < r_o < 0.48$ and showed that the Gibbs ratio for $0 \leq r \leq 0.34$ is given by

$$(5.5) \qquad \frac{2}{\pi}\,\Gamma(r+1)\int_o^{c_1} \mu^{-(r+1)}\, C_{r+1}(\mu)\, d\mu$$

where $C_{r+1}(t)$ is Young's function * defined by

$$(5.6) \qquad C_{r+1}(t) = \frac{t^r}{\Gamma(r)}\int_o^1 (1-\mu)^{r-1} \sin t\mu \, d\mu ,$$

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* Young, W. H., "On infinite integrals involving a generalization of the sine and cosine functions", Quart. Jour. of Math., 43 (1911), 161-177.

and c_1 is the least positive zero of $C_{r+1}(t)$. Szász [104, 1950] obtained the somewhat simpler expression for the GR of $\psi(t)$, proving that

$$(5.7) \qquad GR = \sup \frac{2}{\pi} \int_0^1 (1-\mu)^r \frac{\sin \tau\mu}{\mu} d\mu, \qquad \tau > 0$$

as a special case of his more general result for regular Hausdorff means.

That the Lebesgue constants, $L_n(C, r)$, for the (C, r) means are bounded was first shown by Cramér [18, 1919] who proved the following theorem:

The limit, $L(C, r) = \lim L_n(C, r)$, $(n \to \infty)$, exists as a finite value for every r, $0 < r \leq 1$. Considered as a function of r, this limit always varies in the same sense from $+\infty$ to 1 as r varies from 0 to 1. For a given r it is given by

$$(5.8) \qquad L(C, r) = \frac{2\Gamma(r+1)}{\pi} \int_0^\infty |I_r(\sin x)| \frac{dx}{x^{r+1}}$$

where $I_r(\sin x)$ denotes the integral

$$(5.9) \qquad I_r(\sin x) = \{\Gamma(r)\}^{-1} \int_0^x (x-\mu)^{r-1} \sin \mu \, d\mu.$$

Lorch and Newman [73, 1961] specialized Livingston's formula (10.14) for the Lebesgue constants for regular Hausdorff means to prove the following theorem, avoiding the use of the gamma function:

If $\alpha(t) = 1 - (1 - t)^r$ and $0 < r < 1$, then $\lim L_n(\alpha)$ $(n \to \infty)$ exists and equals

$$(5.10)\qquad L(C, r) = \frac{2}{\pi}\int_0^\infty x^{-2}\left|r\int_0^x (1 - tx^{-1})^{r-1}\sin t\, dt\right|dx$$

$$= \frac{2}{\pi}\int_0^\infty x^{-1}\left|\int_0^1 \sin xt\, d\alpha(t)\right| dx .$$

Moreover, $L(C, r)$ is a non-increasing function of r with $L(C, r) > 1$; also, $L(C, r) \to 1$ as $r \to 1-$ and $L(C, r) \to +\infty$ as $r \to 0+$. For $0 < r_1 < r_2 < 1$, $1 < L(C, r_2) < L(C, r_1)$, and $L(C, r) < L(H, r)$ where $L(H, r)$ is the corresponding limit for Hölder means of the same order.

6. Circle (γ, r) means

Let $\{s_n(t)\}$ be the sequence of partial sums of the series $\sum a_n$. The circle (γ, r) means of this sequence are defined by the transform

$$(6.1)\qquad \sigma_n^*(t) = \sum_n^\infty \binom{\nu}{n} r^{n+1}(1 - r)^{\nu-n} s_\nu(t) .$$

This transform is regular if and only if $0 < r \leq 1$. These means are sometimes called the Taylor means.

Relating his calculations to the series $2\sum \dfrac{\sin(2\nu - 1)t}{2\nu - 1}$, Miracle [80, 1960] proved that for the circle means $\sigma_n^*(t)$ of order $1 - r$ there exists an N such that for all $n \geq N$,

$$(6.2) \qquad |\sigma_n^*(t_n) - \int_0^{\tau} \frac{\sin \mu}{\mu} d\mu| < \epsilon ,$$

where ϵ is an arbitrary, small, positive number, and $\tau = t_n(2n + r)(1 - r)^{-1}$. By taking t_n through the sequence $\{\frac{1 - r}{2n + r} \cdot \pi\}$, it is seen that as n tends to infinity and t_n tends to zero, the Gibbs phenomenon is preserved completely by the (γ, r) means.

Forbes [22, 1962] modified Miracle's proof to the function $\psi(t)$ to show that for this function

$$(6.3) \qquad \sigma_n^*(t_n) = \int_0^{\frac{nt_n}{1-r}} \frac{\sin \mu}{\mu} d\mu + o(1) \qquad n \to \infty ,$$

and again we see that the Gibbs phenomenon is preserved completely.

It might be mentioned that both Miracle and Forbes carried out their work as if for the circle means of order r. However, from their definitions, it is clear that their work is related to these means of order $(1 - r)$.

Apparently unaware of Miracle's work, Ishiguro [44, 1961] obtained essentially the same result. For the series $\psi(t)$, he proved that

$$(6.4) \qquad \lim \sigma_n^*(t_n) = \int_0^{\tau/r} \frac{\sin \mu}{\mu} d\mu, \qquad n \to \infty ,$$

where $nt_n \to \tau$, $0 \leq \tau \leq \infty$, and $nt_n^2 \to 0$ as $t_n \to 0+$. To see that the Gibbs phenomenon is preserved completely, we take t_n through the sequence $\{\frac{r\pi}{n}\}$, r being the order of the circle means.

Ishiguro [43, 1960] also obtained the asymptotic formula for the Lebesgue constants, $L_n(\gamma, r)$, for these means. He proved that

$$(6.5) \qquad L_n(\gamma, r) = \frac{2}{\pi^2} \log \frac{n}{1-r} + a_o + o(1),$$

where a_o is defined by (4.8) or (4.17). The manner of proof is essentially that of Livingston [63, 1954].

Lorch and Newman [75, 1963], motivated by the techniques they used in calculating the Lebesgue constants for the $[F, d_n]$ means, also obtained an independent proof of (6.5).

7. The Euler (ϵ, r) means

These means are defined by the transform

$$(7.1) \qquad \sigma_n(t) = \sum_o^n \binom{n}{\nu} r^\nu (1-r)^{n-\nu} s_\nu(t) .$$

This method is regular if and only if $0 < r \leq 1$.

The Euler (ϵ, r) means are a particular case of the regular Hausdorff means, obtained by taking for the generating function $\alpha(t)$ the function defined by (10.7). They are also a special case of the $[F, d_n]$ means, corresponding to the case where $d_\nu = (1-r)/r$. It turns out that they are the only $[F, d_n]$ means which are also Hausdorff means.

Szász [103, 1950] proved that for these means of the partial sums of $\psi(t)$,

$$(7.2) \qquad \sigma_n(t_n) \to \int_o^{r\tau} \frac{\sin\mu}{\mu}\, d\mu \qquad \text{as } nt_n \to \tau,\ 0 \leq \tau \leq \infty \text{ and } nt_n^2 \to 0 .$$

By choosing $\tau = \pi/r$ and taking t_n through the sequence $\{\frac{\pi}{rn}\}$, it is seen that the Gibbs phenomenon is preserved completely. His other observation, namely, that for the given series and for any sequence $\{t_n\}$, we must have

$$(7.3) \qquad \limsup \sigma_n(t_n) \leq \int_0^\pi \frac{\sin \mu}{\mu} \, d\mu ,$$

follows from (7.2). He also pointed out that the result holds for any function $\sum b_\nu \sin \nu t$ having a discontinuity at $t = 0$, such that

$$(7.4) \qquad f(t) = \sum b_\nu \sin \nu t - c \,.\, \psi(t)$$

is continuous and uniformly summable by Euler means at $t = 0$.

Szász [104, 1950] also specialized his result for the Gibbs ratio for Hausdorff means, given by (10.11), to show that for the (ϵ, r) means,

$$(7.5) \qquad GR = \frac{2}{\pi} \max \int_0^\tau \frac{\sin \mu}{\mu} \, d\mu, \qquad \tau > 0$$

from which it is again clear that the method preserves the Gibbs phenomenon completely.

The Lebesgue constants for the $(\epsilon, 1/2)$ and $(\epsilon, 1/3)$ means were first studied by Prachar and Schmetterer [85, 1948] who showed that in each case, the n^{th} Lebesgue constant, $L_n(\epsilon, r)$, is exactly of order $\log n$ as n becomes infinite. Lorch [67, 1952] obtained the asymptotic expression

$$(7.6) \qquad L_x(\epsilon, 1/2) = \frac{2}{\pi^2} \log x + a_o + O(x^{-1/2})$$

for the $(\epsilon, 1/2)$ means. Here again he used the continuous parameter x in place of n. The constant a_o is defined by (4.17). The manner of proof was to consider

$$(7.7)\qquad L_x(B_4) = \frac{2}{\pi}\int_0^{\pi/2} \exp(-2xt^2)\ \frac{|\sin(2x+1)t|}{\sin t}\,dt$$

$$= L_x(B) + O(x^{-1/2}) \qquad x \to \infty$$

$$= \frac{2}{\pi^2}\log x + a_o + O(x^{-1/2})$$

and comparing it with

$$(7.8)\qquad L_{2x}(\epsilon, 1/2) = \frac{2}{\pi}\int_0^{\pi/2} \cos^{2x} t\ \frac{|\sin(2x+1)t|}{\sin t}\,dt.$$

Livingston [63, 1954] solved the problem for the (ϵ, r) means completely by proving that for these means, the Lebesgue constants are given by

$$(7.9)\qquad L_n(\epsilon, r) = \frac{2}{\pi^2}\log\frac{nr}{1-r} + a_o + o(1) \qquad n \to \infty$$

where again a_o is defined by (4.17). He first proved that for the regular Hausdorff means corresponding to the weight function $\alpha(t)$, the Lebesgue constants are given by

$$(7.10)\qquad L_n(\alpha) = \frac{2}{\pi}\int_0^{\pi/2} \Big|\frac{1}{x}\int_0^{1-} \{1 - 4t(1-t)\sin^2 x\}^{n/2} \sin 2nxt\,d\alpha(t)$$

$$+ \{\alpha(1) - \alpha(1-)\}\frac{\sin(2n+1)x}{\sin x}\Big|\,dx + o(1), \qquad n \to \infty.$$

For $\alpha(t)$ defined by (10.7) for the (ϵ, r) means, he reduced this to

$$(7.11)\qquad L_n(\epsilon, r) = L_{n^*}(\epsilon, 1/2) + o(1)$$

where $n' = nr/(1 - r)$, and $L_{n'}(\epsilon, 1/2)$ is the $\{nr/(1 - r)\}^{th}$ Lebesgue constant given by (7.6). The result (7.9) is now apparent.

It might be mentioned that to date there is some lack of uniformity in the notation relating to the (ϵ, r) method. Thus, the Euler means of order r are denoted by (ϵ, r) by Ishiguro [44] and by the analogue of $(E, r/(1 - r))$ by Lorch [67]. Occasionally also the notation (E_r) is used. For the sake of consistency, these have been changed to the (ϵ, r) notation in this paper.

8. $[F, d_n]$ means

For the sequence $\{s_n(t)\}$, the $[F, d_n]$ means are defined by

$$(8.1) \qquad \sigma_n(t) = \sum_{0}^{n} P_{n\nu} s_\nu(t)$$

where $P_{oo} = 1$, and for $d_\nu \geq 0$, $n = 1, 2, \ldots$, the constants $P_{n\nu}$ are given by

$$(8.2) \qquad \prod_{1}^{n} \frac{x + d_\nu}{1 + d_\nu} = \sum P_{n\nu} x^\nu .$$

The condition for regularity is that $\sum (1 + d_\nu)^{-1}$ diverges, or equivalently $\sum' d_n^{-1}$ diverges, the prime indicating summation over non-zero elements d_ν. The $[F, d_n]$ means were studied by Jakimovski [52] as generalizations of the Euler (ϵ, r) means, to which they reduce for $d_\nu = (1 - r)/r$, and of the Lototsky means to which they reduce for $d_\nu = (\nu - 1)/m$.

The Gibbs phenomenon for the regular $[F, d_n]$ means was studied by Miracle [80, 1960] who proved that this method preserves the Gibbs phenomenon completely. He related his calculations to the partial sums of the series $2\sum \frac{\sin(2\nu - 1)\,t}{2\nu - 1}$.

In the calculation of the Lebesgue constants, Lorch and Newman [74, 1962] considered two cases, one in which $s_n = 2\sum_1^n d_\nu(1+d_\nu)^{-2}$ is bounded, and the other in which it is unbounded. Corresponding to these they obtained, respectively, the Lebesgue constants

$$(8.3) \qquad L_n(F, d_n) = \frac{4}{\pi^2} \log \mu_n + O(1)$$

$$(8.4) \qquad L_n(F, d_n) = \frac{2}{\pi^2} \log \frac{\mu_n^2}{2s_n} + a_o + o(1) .$$

The constant a_o is again the constant defined by (4.17) and $\mu_n = 1 + 2\sum_1^n (1 + d_\nu)^{-1}$. In each case, it is seen that the Lebesgue constants are unbounded. Setting $d_\nu = (1 - r)/r$ to obtain the Euler (ϵ, r) means, the authors derived Livingston's formula (7.9), and setting $d_\nu = m^{-1}(\nu - 1)$, they obtained the Lebesgue constants $L_n(S, m)$ given by (15.2) for the Lototsky (S, m) means.

9. Harmonic means

For the sequence $\{s_n(t)\}$, the harmonic means are defined by the transform

$$(9.1) \qquad \sigma_n(t) = (\log n)^{-1} \sum_1^n \frac{s_{n-\nu}(t)}{\nu} .$$

The method is regular. For the Gibbs phenomenon for these means, Hsiang [36, 1962] proved the following theorem:

> The Gibbs phenomenon for the harmonic means of the sequence of partial sums of $\psi(t)$ presents itself at the point $t = 0$ and has the Gibbs ratio

$$GR = \frac{2}{\pi}\int_0^{\pi} \frac{\sin \mu}{\mu}\, d\mu . \tag{9.2}$$

10. Hausdorff means

Let p be a matrix whose elements are $p_{m,n} = (-1)^n \binom{m}{n}$ for $0 \leq n \leq m$, and $p_{m,n} = 0$ for $m < n$, where m, n are positive integers or zero. Let μ be a matrix with the elements $\mu_{m,n} = \mu_n$ for $m = n$, and $\mu_{m,n} = 0$ otherwise. Corresponding to the sequence $\{s_n(t)\}$, let s be a matrix with the elements $s_n(t)$ in the first column and zeros elsewhere, and let $h_n(t)$ denote the elements in the first column of the matrix h, where

$$h = p\mu p^{-1} s . \tag{10.1}$$

The matrix $p\mu p^{-1}$ is a Hausdorff matrix corresponding to the sequence $\{\mu_n\}$, and the sequence $\{s_n(t)\}$ is said to be summable to the sum $s(t)$, in the Hausdorff sense, corresponding to the sequence $\{\mu_n\}$, if the sequence $\{h_n(t)\}$ converges to $s(t)$ as n approaches infinity.

For the proof of the following theorem, the reader is referred to Widder [121, p. 119]:

The Hausdorff method of summability corresponding to the sequence $\{\mu_n\}$ is regular if and only if $\{\mu_n\}$ is a moment sequence; that is,

$$(10.2) \qquad \mu_n = \int_0^1 x^n \, d\alpha(x) \qquad n = 0, 1, 2, \ldots$$

where $\alpha(x)$ is of bounded variation in $(0, 1)$ $\alpha(0) = \alpha(0+) = 0$, and $\alpha(1) = 1$.

Using this theorem, the definition of regular Hausdorff means may be put into the more compact form in which it is more generally used. Let $p\mu p^{-1} = (\ell_{n\nu})$ be the matrix with the elements $\ell_{n\nu}$. Setting $s_\nu(t) = t^\nu$, and using the fact that the matrix p is its own inverse, we get

$$(10.3) \qquad \begin{aligned} h_n(t) &= \sum \ell_{n\nu} \, s_\nu(t) \\ &= \sum_0^n \ell_{n\nu} \, t^\nu \\ &= \sum_0^n (-1)^j \binom{n}{j} \mu_j \sum_0^j (-1)^\nu \binom{j}{\nu} t^\nu \\ &= \sum_0^n (-1)^j \binom{n}{j} \int_0^1 x^j (1 - t)^j \, d\alpha(x) \\ &= \int_0^1 \sum_0^n (-1)^j \binom{n}{j} (x - xt)^j \, d\alpha(x) \\ &= \int_0^1 (1 - x + xt)^n \, d\alpha(x) \\ &= \sum_0^n \binom{n}{\nu} t^\nu \int_0^1 x^\nu (1 - x)^{n-\nu} \, d\alpha(x) \end{aligned}$$

Comparing the coefficients of like powers of t, we have

$$(10.4) \qquad \ell_{n\nu} = \binom{n}{\nu} \int_0^1 x^\nu (1 - x)^{n-\nu} \, d\alpha(x)$$

$$(10.5) \qquad h_n(t) = \sum_0^n \binom{n}{\nu} s_\nu(t) \int_0^1 x^\nu (1 - x)^{n-\nu} \, d\alpha(x) .$$

(10.5) is the form in which the regular Hausdorff transform is often defined.

The regular Hausdorff method is a generalization of the Cesáro (C, r) means, the Euler (ϵ, r) means and the Hölder (H, r) means, which are obtained, respectively, by taking

$$(10.6) \qquad \alpha(x) = 1 - (1 - x)^r$$

$$(10.7) \qquad \begin{aligned} \alpha(x) &= 0 \quad \text{for} \quad 0 \leq x < r \\ &= 1 \quad \text{for} \quad r \leq x \leq 1 \end{aligned}$$

$$(10.8) \qquad \alpha(x) = (\Gamma(r))^{-1} \int_0^x (\log \frac{1}{y})^{r-1} \, dy .$$

For the Gibbs phenomenon for the Hausdorff means of the partial sums of $\psi(t)$, Szász [104, 1950] proved the following theorem:

If $nt_n \to \tau \leq \infty$, then the Hausdorff means of $\psi(t)$,

$$(10.9) \qquad h_n(t_n) \to \int_0^1 \int_0^\tau \frac{\sin \mu x}{x} \, dx \, d\alpha(\mu) , \qquad n \to \infty .$$

For $\tau < \infty$,

$$(10.10) \qquad h_n(t_n) \to \int_0^1 \{1 - \alpha(\mu)\} \frac{\sin \tau\mu}{\mu} d\mu, \qquad n \to \infty,$$

where $\alpha(\mu)$ is the Hausdorff weight function.

From this result he obtained the Gibbs ratio

$$(10.11) \qquad GR = \frac{2}{\pi} \max \int_0^1 \{1 - \alpha(\mu)\} \frac{\sin \tau\mu}{\mu} d\mu,$$

and the theorem

For the Hausdorff means $h_n(t)$,

$$(10.12) \qquad \limsup_{\substack{n \to \infty \\ t \to 0}} h_n(t) = \max \int_0^1 \{1 - \alpha(\mu)\} \frac{\sin \tau\mu}{\mu} d\mu, \qquad \tau \to 0 ;$$

if this maximum is attained for $\tau = \tau'$, then

$$(10.13) \qquad \limsup_{\substack{n \to \infty \\ t \to 0}} h_n(t) = \lim h_n(t_n) \qquad nt_n \to \tau' .$$

Szász specialized the above results to the Cesáro, Euler, and Hőlder methods. For a discussion of these, the reader is referred to the pertinent sections of this chapter.

Livingston [62, 1953] studied the influence of the jump discontinuities of the weight function $\alpha(t)$ on Gibbs phenomenon. He proved that $h_n(t)$ exhibits the Gibbs phenomenon whenever $\alpha(t)$ is a regular Hausdorff step function which has precisely two jumps, or whose points of jump are linearly independent over the rationals. The manner of proof was such that it could not be applied to step function kernels outside these two categories.

Newman [83, 1962] generalized Livingston's result by proving the following theorem:

> Let $\alpha(t)$, such that $\alpha(0) = \alpha(0+)$ and $\alpha(1) - \alpha(0) = 1$, have at least one jump discontinuity, and let $\int_0^1 \mu^{-2}|d\alpha(\mu)| < \infty$. Then the Gibbs phenomenon occurs for the regular Hausdorff means corresponding to the weight function $\alpha(t)$.

Newman also proved that this theorem is not true if the condition $\int_0^1 \mu^{-2} |d\alpha(\mu)| < \infty$ is replaced by $\int_0^1 \mu^{-1/2}|d\alpha(\mu)| < \infty$, and raised the problem of finding an exponent k_o such that the theorem would be true for $\int_0^1 \mu^{-k}|d\alpha(\mu)| < \infty$, $k \geq k_o$, but false for $k < k_o$. Livingston [64, 1963] returned to this problem and proved that the Gibbs phenomenon exists if $\int_0^1 \mu^{-2} \alpha(\mu)\, d\mu$ exists, perhaps as an improper Riemann integral, and if $y^{-1} \int_0^y |\cos x \int_0^1 \mu^{-2} \alpha(\mu)\, d\mu + \int_0^1 \cos x\mu \{\mu^{-1} d\alpha(\mu)\}|dx \geq m > 0$ for all large y; and in particular if $\int_0^1 \mu^{-1} |d\alpha(\mu)| < \infty$. Combining this with Newman's result, we have $1/2 < k_o \leq 1$.

For the Lebesgue constants, $L_n(\alpha)$, Livingston [63, 1954] proved that

$$(10.14) \qquad L_n(\alpha) = \frac{2}{\pi}\int_0^{\pi/2} \Big|x^{-1}\int_0^{1-}\{1-4\mu(1-\mu)\sin^2 x\}^{n/2}\sin 2nx\mu\, d\alpha(\mu) + \{\alpha(1)-\alpha(1-)\}\frac{\sin(2n+1)x}{\sin x}\Big|dx + o(1), \quad n \to \infty.$$

Starting with this result, Lorch and Newman [73, 1961] proved the following theorem:

Let $L_n(\alpha)$ denote the n^{th} Lebesgue constant for the regular Hausdorff method with weight function $\alpha(t)$. Then

$$(10.15) \qquad L_n(\alpha) = C(\alpha) \log n + o(\log n) \qquad n \to \infty$$

where

$$(10.16) \qquad C(\alpha) = \frac{2}{\pi^2} |\alpha(1) - \alpha(1-)| + \frac{1}{\pi} M\left\{\left| \sum \{\alpha(\xi_\nu{}^+) - (\xi_\nu{}^-)\} \sin \xi_\nu t \right|\right\} .$$

Here ξ_ν is the ν^{th} discontinuity (jump) of $\alpha(t)$ and the summation extends over all such (possibly countably infinite) values. $M\{f(t)\}$ represents the mean value of the almost periodic function $f(t)$. Furthermore,

$$(10.17) \qquad 0 \leq C(\alpha) \leq \frac{4}{\pi^2} \cdot V(\alpha)$$

where $V(\alpha)$ is the total variation of $\alpha(t)$, $0 \leq t \leq 1$, and

$$(10.18) \qquad C(\alpha) = 0$$

if and only if $\alpha(t)$ is continuous. If, in addition, the method is totally regular so that $V(\alpha) = 1$, then also

(10.19) $\quad C(\alpha) = 4/\pi^2$ if and only if the method is ordinary convergence;

(10.20) $\quad C(\alpha) \leq 2/\pi^2$ when $\alpha(1-) = \alpha(1)$, and in this case,

(10.21) $\quad C(\alpha) = 2/\pi^2$ if and only if the method is of Euler type.

The manner of proof of (10.15) was to start with Livingston's formula (10.14) and obtain a new estimate

$$(10.22) \qquad L_n(\alpha) = \frac{2}{\pi}\int_1^{n^{1/2}} |x^{-1}\int_0^1 \sin x\mu d\alpha(\mu)|dx + \frac{2}{\pi^2}|\alpha(1) - \alpha(1-)| \log n + o(\log n).$$

Breaking $\alpha(t)$ into a continuous component $h(t)$ and the discontinuous component

$$(10.23) \qquad j(t) = \sum \{\alpha(\xi_\nu^+) - \alpha(\xi_\nu^-)\}, \qquad \xi_\nu < t,$$

where the set $\{\xi_\nu\}$ consists of all the points of discontinuity of $\alpha(t)$, the authors first showed that

$$(10.24) \qquad \int_1^n x^{-1} |\int_0^1 \sin x\mu d\alpha(\mu)|dx = O(1),$$

and then, by suitable substitutions, obtained (10.15). The remaining results in the theorem follow, with the 'if' portion of (10.21) following from Livingston's work [63], and the 'only if' from a lemma which the authors prove. The error term in (10.15) is proved to be best possible.

The authors draw some interesting observations from their theorem. Equation (10.18) shows that any Hausdorff method with a discontinuous weight function exhibits the du Bois-Reymond and the Lebesgue singularities. Equations (10.19) and (10.20) show, respectively, that among all totally regular Hausdorff methods, ordinary convergence has the

maximum principal term for the Lebesgue constants, and that the Euler methods possess the same extremal property in the class of totally regular Hausdorff methods with weight function continuous at $t = 0$. They also point out that equation (10.18) does not imply that a Hausdorff method with a continuous, or even an absolutely continuous, weight function, sums the Fourier series of a continuous function everywhere, since the remainder in (10.15) can be unbounded.

11. Quasi-Hausdorff means

These means are defined by the transform

$$(11.1) \qquad h_n^*(t) = \sum_n^\infty \binom{\nu}{n} s_\nu(t) \int_0^1 x^{n+1}(1-x)^{\nu-n}\, d\alpha(x).$$

The transform is regular if and only if $\alpha(1) - \alpha(0+) = 1$, where $\alpha(x)$ is of bounded variation in $0 \le x \le 1$. Now for $\alpha(x)$ defined by (10.7), the Hausdorff transformation reduces to the Euler (ϵ, r) transform defined by (7.1). The same function $\alpha(x)$ yields the circle (γ, r) means defined by (6.1) when applied in the quasi-Hausdorff transform (11.1).

Ishiguro and Kuttner [43, 1963] generalized Ishiguro's earlier result for the circle (γ, r) means [44, 1961] by proving the following theorem:

For the regular quasi-Hausdorff means of $\psi(t)$, we have

$$(11.2) \qquad \lim h_n^*(t_n) = \int_0^1 \int_0^{\tau/r} \frac{\sin\mu}{\mu}\, d\mu\, d\alpha(r), \qquad n \to \infty,$$

provided that the weight function $\alpha(r)$ is continuous at $r = 0$, $nt_n \to \tau$ and $nt_n^2 \to 0$.

Ishiguro's earlier result for the circle means then follows immediately from this theorem.

Corresponding to the investigations of Livingston and Newman pertaining to the influence of the jump discontinuities of the weight function $\alpha(t)$ on the Gibbs phenomenon in the case of regular Hausdorff means, Ishiguro [46, 1964] proved the following theorem for the quasi-Hausdorff means:

> If $\alpha(t)$ is a step function which is continuous at $t = 0$, then the quasi-Hausdorff means of the partial sums of the series $\psi(t)$ exhibit the Gibbs phenomenon.

Setting $\alpha(t) = h(t) + j(t)$, where $h(t)$ is continuous and $j(t) = \sum \{\alpha(\xi_\nu^{+}) - \alpha(\xi_\nu^{-})\}$, $\xi_\nu \leq t$, he generalized the above result to the following:

> If $\alpha(t)$ is a weight function which has at least one jump discontinuity, but is continuous at $t = 0$, and if its continuous part $h(t)$ satisfies
>
> $$\int_1^\infty \mu^{-1} \{h(\mu^{-1}) - \mu^{-1}h(1)\} \sin \tau\mu d\mu \in L'(1, \infty), \quad 0 < \tau < \infty,$$
>
> then the regular quasi-Hausdorff means of $\psi(t)$ exhibit the Gibbs phenomenon.

For the Lebesgue constants, Ishiguro [47, 1964] followed the manner of proof of Lorch and Newman [73, 1961] for the Hausdorff means to prove the corresponding theorem for the quasi-Hausdorff methods:

If the weight function $\alpha(t)$ is a step function which is continuous at the origin, then

$$(11.3) \qquad L_n^*(\alpha) = C^*(\alpha) \log n + o(\log n), \qquad n \to \infty ,$$

where

$$(11.4) \qquad C^*(\alpha) = \frac{2}{\pi^2}|\alpha(1)-\alpha(1-)| + \frac{1}{\pi} M\left\{\left|\sum \{\alpha(\xi_\nu{}^+)-\alpha(\xi_\nu{}^-)\}\sin\frac{t}{\xi_\nu}\right|\right\} ,$$

where ξ_ν is the ν^{th} discontinuity (jump) of $\alpha(t)$ and the summation extends over all such (possibly countably infinite) values. $M\{f(t)\}$ represents the mean value of the almost periodic function $f(t)$.

12. Hölder (H, r) means

The Hölder (H, r) mean of integral order r may be defined as an iterated arithmetic mean. Let $\{s_n(t)\}$ be a given sequence and set

$$(12.1) \qquad H_n^{(1)} = \frac{s_o + s_1 + \ldots + s_n}{n+1}$$

$$(12.2) \qquad H_n^{(r)} = \frac{H_o^{(r-1)} + H_1^{(r-1)} + \ldots + H_n^{(r-1)}}{n+1} .$$

As mentioned earlier, this method is the regular Hausdorff method corresponding to the weight function defined by (10.8). The extension to non-integral order r is again apparent. A sequence $\{s_n(t)\}$ is said to be summable by Hölder's method of order r to sum $s(t)$ if

$$(12.3) \qquad \lim H_n^{(r)} = s(t) \qquad n \to \infty .$$

Szász [104, 1950] studied Gibbs phenomenon for these means as a particular case of his results for the Hausdorff means. He proved that there exists a constant γ such that the Gibbs phenomenon exists for the (H, r) means of $\psi(t)$ for $r < \gamma$, but not for $r \geq \gamma$, where $0.5826 < \gamma < 0.5850$. Note that the corresponding constant r_o for the (C, r) means is 0.4395... .

For the Lebesgue constants, Lorch and Newman [73, 1961] proved the following theorem:

If $\alpha(t)$ is given by (10.8) for the Hölder means of the regular Hausdorff methods, and $0 < r < 1$, then $\lim L_n(\alpha)$ exists as $n \to \infty$ and equals

$$(12.4) \qquad L(H, r) = \frac{2}{\pi} \int_0^\infty x^{-2} |\{\Gamma(r)\}^{-1} \int_0^x (\log x\mu^{-1})^{r-1} \sin \mu \, d\mu | dx.$$

Moreover,

$$(12.5) \qquad L(C, r) \leq L(H, r)$$

and for $0 < r_1 < r_2 < 1$,

$$(12.6) \qquad 1 < L(H, r_2) \leq L(H, r_1),$$

with $L(H, r) \to 1$ as $r \to 1-$, and $L(H, r) \to +\infty$ as $r \to 0+$.

The main statement of the theorem follows by estimating Livingston's integral (10.14) for the specialized form of $\alpha(t)$. The rest of it follows from the lemma

Given regular Hausdorff methods T_1 and T_2 with associated Lebesgue constants $L_n(T_1)$ and $L_n(T_2)$, respectively. Suppose that there exists a totally regular Hausdorff method u such that $T_2 = uT$. Then, if $\lim L_n(T_i)$ exists, $n \to \infty$, and equals $L(T_i)$, $i = 1, 2$, we have $L(T_2) \leq L(T_1)$.

which the authors prove, using the known property that the matrix u has essentially non-negative elements *.

13. K and K' - methods

Let a regular, linear transformation of the sequence $\{s_n(t)\}$ be defined by

$$\sigma_n(t) = \sum k_{n\nu} s_\nu(t). \tag{13.1}$$

If, in addition,

$$\sum \nu |k_{n\nu}| < M < \infty \quad \text{for all } n, \tag{13.2}$$

then the method is said to be a K-method [34, p. 56].

Kuttner [58] considered a linear series to sequence transformation, which he called the K'-method, defined by

$$\sigma_n^*(t) = \sum k_{n\nu} a_\nu(t). \tag{13.3}$$

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* See Hurwitz, W. A., Some properties of methods of evaluation of divergent sequences. Proc. London Math. Soc. (2) 26 (1927), 231-248.

The regularity conditions for this method are

$$(13.4) \qquad \sum |k_{n,\nu} - k_{n,\ \nu+1}| < M < \infty \qquad n = 0, 1, 2, \ldots$$

$$\lim k_{n\nu} = 1 \qquad n \to \infty; \quad \nu = 0, 1, 2, \ldots$$

$$\sum |k_{n\nu}| < \infty \qquad n = 0, 1, 2, \ldots$$

As a special case of this method, he considered the method where the sum $\sum a_\nu$ is defined by

$$(13.5) \qquad \lim \sum q(\tfrac{\nu}{\mu})\, a_\nu \qquad \mu \to \infty$$

where $q(x)$ is continuous, absolutely integrable and of bounded variation in $(0, \infty)$, and $q(0) = 1$. This is a generalization of the Riesz (R, n^λ, κ), the Abel (A, n^λ) and the Riemann $(R, 2)$ methods, which are obtained, respectively, by setting

$$(13.6) \qquad q(x) = (1 - x^\lambda)^\kappa \qquad x \le 1$$

$$= 0 \qquad x > 1$$

$$(13.7) \qquad q(x) = \exp(-x^\lambda)$$

$$(13.8) \qquad q(x) = (\sin^2 x)/x^2.$$

For the K-methods, Kuttner [57, 1945] proved the following interesting result:

In order that a given K-method should be such that the Gibbs phenomenon does not occur, that is,

$$(13.9) \qquad \liminf_{t \to t_o} f(t) \leq \liminf_{\substack{t \to t_o \\ n \to \infty}} \sigma_n(t) \leq \limsup_{\substack{t \to t_o \\ n \to \infty}} \sigma_n(t)$$

$$\leq \limsup_{t \to t_o} f(t),$$

it is necessary and sufficient that the kernel $K_n(t)$ should be bounded below for all n, t.

In a subsequent paper, Kuttner [58, 1947] extended his work to the K' methods, and specifically to those K' methods for which the generalized sum is defined by

$$(13.10) \qquad \lim_{\mu \to \infty} \sum q(\tfrac{\nu}{\mu}) a_\nu,$$

where $q(x)$ is continuous, absolutely integrable, and of bounded variation in $(0, \infty)$, and $q(0) = 1$. Kuttner proved that if the kernels of such methods are bounded below, then they are non-negative. His earlier result, then, implies that for any K' method which is also a K method, a necessary and sufficient condition that the Gibbs phenomenon should not occur is that the kernels be positive. The Riesz (R, n^λ, κ), Abel (A, n^λ) and Riemann (R, 2) methods are in this category.

14. The Lambert means

The series $\sum a_\nu$ is said to be summable (L) to s, or

$$(14.1) \qquad \sum_1^n a_\nu = s_n \to s, \qquad n \to \infty,$$

if

$$(14.2)\qquad F(t) = \sum a_\nu \frac{nt\, e^{-nt}}{1 - e^{-nt}} \to s \qquad t \to 0+ .$$

Szász [107, 1952] proved that the Gibbs ratio, GR, for the Lambert transform of the series $\psi(t)$ satisfies the inequality GR $\leq$ 1, and hence this transform presents no Gibbs phenomenon.

15. Lototsky means*

Let $\{s_k(t)\}$ be the sequence of partial sums of the series $\sum a_n(t)$. For each $n = 1, 2, 3, \ldots$, let $p_n(x)$ be the polynomial

$$(15.1)\qquad p_n(x) = x(x+1)(x+2) \ldots. (x+n-1)$$

$$= p_{n1}x + p_{n2}x^2 + \ldots + p_{nn}x^n.$$

For the $p_{n\nu}$ thus defined, the Lototsky means are defined by the transform

$$(15.2)\qquad \sigma_n(t) = \sum_1^n \frac{p_{n\nu}}{n!} s_\nu(t).$$

The series $\sum a_n(t)$ is said to be summable Lototsky if $\lim \sigma_n(t)$ exists as $n \to \infty$. The method is regular.

---

* Agnew, R. P., "The Lototsky method for evaluation of series". Mich. Math. Jour., 4 (1957), 105-128.

Miracle [80, 1960] proved that the $[F, d_n]$ means preserve the Gibbs phenomenon completely. Since the Lototsky transform is obtained by specializing the $[F, d_n]$ transform, it follows that the result applies to the Lototsky transform.

The Lebesgue constants, $L_n(S, m)$, for this method were obtained by Lorch and Newman [74, 1962] who proved that

$$(15.2) \qquad L_n(S, m) = \frac{2}{\pi^2} \log \log n^2 + a_o + \frac{2}{\pi^2} \log \frac{2}{m^3} + o(1),$$

where a_o is the constant defined by (4.8) or (4.17). They obtained this representation by specializing their result for the $[F, d_n]$ means.

16. <u>Nörlund (N, p_n) means</u>

For a given sequence $\{p_n\}$ of numbers such that $p_n \geq 0$, $p_o > 0$, let $P_n = \sum_o^n p_\nu$. The n^{th} Nörlund mean is defined by the transform

$$(16.1) \qquad \sigma_n(t) = P_n^{-1} \sum_o^n p_{n-\nu} s_\nu(t).$$

If $\sigma_n(t)$ approaches a limit $\sigma(t)$ as $n \to \infty$, the sequence $\sum a_\nu(t)$ is said to be summable (N, p_n). The transform is regular if and only if $\lim p_n/P_n = 0, \quad n \to \infty$.

For $p_n = \binom{n + r - 1}{r - 1} = \frac{\Gamma(n + r)}{\Gamma(n + 1)\Gamma(r)}$ $(r > 0)$, the Nörlund (N, p_n) means reduce to the Cesáro (C, r) means.

For the Nörlund (N, p_n) method, Prasad and Siddiqi [86, 1958] proved that if $p_n > 0$ and monotone decreasing, then the (N, p_n) means of the partial sums of the Fourier series of any function $f(t)$ must fail to exhibit the Gibbs phenomenon. They also established the following condition between two Nörlund transforms under which the non-occurrence of the Gibbs phenomenon for one method will imply its non-occurrence for the other:

> Let (N, p_n) and (N, q_n) be two regular Norlund methods with positive coefficients. Let $\{k_\nu\}$ be defined by
>
> $$(16.1) \quad \sum k_\nu z^\nu = \left\{\sum q_\nu z^\nu\right\} \Big/ \left\{\sum p_\nu z^\nu\right\} = \left\{\sum Q_\nu z^\nu\right\} \Big/ \left\{\sum P_\nu z^\nu\right\},$$
>
> such that $k_\nu \geq 0$ for all ν and $\sum_o^n k_\nu = o(Q_n), \quad n \to \infty.$
>
> If the sequence $\{s_n(t)\}$ is bounded in the neighbourhood of $t = t_o$, then the non-persistence of the Gibbs phenomenon for the (N, p_n) means necessarily implies the same for the (N, q_n) means.

17. Perron means

The summation of the series $\sum a_\nu$ by the method of Perron is defined by the transform

$$(17.1) \qquad \sigma_r(x) = \sum a_\nu \frac{\Gamma(x + r)\ \Gamma(x + \nu)}{\Gamma(x)\ \Gamma(x + r + \nu)}$$

$$= \{B(r, x)\}^{-1} \sum a_\nu\ B(r, x + \nu)$$

$$= \{B(r, x)\}^{-1} \sum a_\nu \int_0^1 t^{x+\nu-1}(1-t)^{r-1}\,dt$$

$$= \{B(r, x)\}^{-1} \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+r}} \sum a_\nu \left(\frac{t}{1+t}\right)^\nu dt$$

where the last result follows by a change of variable. $B(m, n)$ is the beta function

$$(17.2) \qquad B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} .$$

The generalized sum of the series $\sum a_\nu$ is defined as the limit of $\sigma_r(x)$ as $x \to \infty$, whenever the limit exists.

de Sz. Nagy [112] used another method of summability due to Perron, which will be referred to as method B, in which the generalized sum of a series $\sum a_\nu$ is defined as the limit of

$$(17.3) \qquad \sum_0^{n-1} \varphi\left(\frac{\nu}{n}\right) a_\nu , \qquad n \to \infty ,$$

where $\varphi(x)$ is absolutely continuous in $(0, 1)$, its derivative $\varphi'(x)$ is of bounded variation except in the neighbourhood of a finite number of points, and $\varphi(0) = 1$. (Compare Kuttner's K' method.)

Szász [105, 1950] proved that for the series $\psi(t)$, the Perron transform $\sigma_r(t)$ does not exceed $\pi/2$, and therefore this method does not exhibit the Gibbs phenomenon for any $r > 0$.

Corresponding to Perron's method B, if $\varphi(t)$, the function defining the transform (17.3), is absolutely continuous in (0, 1), and its derivative $\varphi'(t)$ is of bounded variation except possibly in the neighbourhood of a finite number of points, de Sz. Nagy [112, 1948] proved that the Lebesgue constants $L_n(\varphi)$ satisfy the inequality

$$(17.4) \qquad L_n(\varphi) \leq |\varphi'(0+)| + \frac{2}{\pi} \int_{0+}^{1-} (1-\mu)\{2+\log \frac{1+\mu}{1-\mu}\} |d\varphi'(\mu)| .$$

He also derived a similar representation for the conjugate Lebesgue constants $\overline{L_n(\varphi)}$.

If $\varphi(t)$ is decreasing and convex, he proved that

$$(17.5) \qquad L_n(\varphi) = 1$$

$$\overline{L_n(\varphi)} = \frac{4}{\pi} \sum_{o}^{N} (2\nu - 1)^{-1} \varphi(\frac{(2\nu - 1)}{n}) \qquad N = [1/2(n-1)]$$

18. Riemann (R, k) means

The Riemann (R, k) summation method for the series $\sum a_\nu$ is defined by

$$(18.1) \qquad \sigma_h(t) = \sum a_\nu(t) \left(\frac{\sin \nu h}{\nu h}\right)^k$$

where k is a positive integer. The method is regular if $k > 1$, but not if $k = 1$. The generalized sum is taken as the limit of the right hand side as h tends to zero.

The Riemann (R, k) method is a specialization of the K' methods considered by Kuttner [58, 1947]. His results for this general

class of methods imply that for the (R, k) means, the Gibbs phenomenon cannot occur whenever the kernel

$$K(h, t) = 1/2 + \sum \left(\frac{\sin \nu h}{\nu h}\right)^k \cos \nu t \qquad h \to 0$$

is positive.

Lee [60, 1956] proved that if a function $f(t)$ is of bounded variation and has a jump at $t = t_o$, then the Gibbs set corresponding to the (R, k) transform is the closed interval of length $|f(t_o+) - f(t_o-)|$, with centre at $1/2\{f(t_o+) + f(t_o-)\}$. This result depends on the uniform convergence with respect to h of the (R, k) transform of the Fourier series of $f(t)$ at every point of continuity of $f(t)$. Thus it is seen that these means do not present the Gibbs phenomenon.

19. The (R_1) and (R_2) means

These are closely related to the Riemann (R, k) means. Each is a sequence to sequence transform. The (R_1) method is defined by the transform

$$(19.1) \qquad \sigma_h^1(t) = \sum s_\nu(t)\, \nu^{-1} \sin \nu h$$

and the (R_2) method by

$$(19.2) \qquad \sigma_h^2(t) = \frac{2}{\pi} \sum s_\nu(t) \frac{\sin^2 \nu h}{n^2 h} .$$

In each case, the generalized sum of the series $\sum a_\nu$ is defined by the limit of the right hand side as h tends to zero, if this limit exists. These means are regular.

That the R_1 method does not present the Gibbs phenomenon and that the Lebesgue constants $L_n(R_1)$ are uniformly bounded was proved by Szász [106, 1951].

20. Riesz (R, λ, κ) means

For a sequence of numbers $0 \leq \lambda_o < \lambda_1 < \lambda_2 < \ldots , \lambda_n \to \infty$, set

$$(20.1) \qquad R_\lambda(x) = \sum_o^n a_\nu = s_n \qquad \lambda_n < x \leq \lambda_{n+1}$$
$$= 0 \qquad x \leq \lambda_o .$$

For $\kappa > 0$, suppose that

$$(20.2) \qquad R_\lambda^\kappa(\omega) = \frac{\kappa}{\omega^\kappa} \int_o^\omega R_\lambda(t)(\omega - t)^{\kappa-1}\, dt$$

tends to a limit s as ω tends to infinity. Then $\sum a_\nu$ is said to be summable (R, λ, κ) to s.

Now by partial integration,

$$(20.3) \qquad R_\lambda^\kappa(\omega) = \int_o^\omega (1 - \frac{t}{\omega})^\kappa \, d\, R_\lambda(t)$$
$$= \sum_{\lambda_\nu < \omega} (1 - \frac{\lambda_\nu}{\omega})^\kappa a_\nu .$$

(20.3) defines Riesz's typical means. If $\lambda_\nu = \nu$ and ω is a positive integer n, the means are called Riesz's arithmetic means. Similarly, if ω is a positive number n^2, and $\lambda_\nu = \nu^2$, the means are called Riesz's circular means. These means are regular.

The method defined by Riesz's circular means is often called Bochner's summation method.

For the (R, n^{λ}, κ) means of the Fourier series of a function having a simple discontinuity, Kuttner [56, 1944] proved that, for $0 < \lambda < 2$, there exists a continuous, strictly increasing function $r(\lambda)$ such that the Gibbs phenomenon vanishes for $\kappa \geq r(\lambda)$, but not for $\kappa < r(\lambda)$. This function tends to zero as λ tends to zero, equals Cramér's constant r_o when $\lambda = 1$, and tends to infinity as $\lambda \to 2$. If $\lambda = 2$, the Gibbs phenomenon persists however large κ might be.

For $\lambda = 2$, Cheng [10, 1950] proved that the Gibbs set corresponding to the (R, n^{λ}, κ) means of the Fourier series of a function $f(t)$ having a simple discontinuity is a closed interval of length $|\{2dP_{\kappa}\}/\pi|$, symmetric about $1/2\{f(t+) + f(t-)\}$, where

$$d = f(t+) - f(t-)$$

$$P_{\kappa} = (\pi/2)^{1/2} 2^{\kappa} \Gamma(\kappa + 1) \int_0^{j_{\kappa}} J_{\kappa+1/2}(\mu) \mu^{-(\kappa+1/2)} d\mu$$

$$> \pi/2,$$

and j_{κ} is the least positive zero of the Bessel function $J_{\kappa+1/2}(\mu)$.

Now suppose that $\lambda = \lambda(\omega) = \exp \mu(\omega)$, and that a) $\mu(\omega)$ is differentiable and monotone increasing in $(0, \infty)$ and $\mu(\omega) \to \infty$ as $\omega \to \infty$, b) $\mu'(\omega)$ is monotone decreasing for $\omega > A$, and $\mu'(\omega) \to 0$, $\omega\mu'(\omega) \to \infty$ as $\omega \to \infty$, and c) $\lambda'(\omega)$ is monotone increasing for $\omega > A$. Under these conditions, Matsumoto [78, 1956] proved the theorem

If we let $L_R(\omega)$ denote the Lebesgue constant of the $(R, \lambda(\omega), \kappa)$ summation, then

$$L_R(\omega) \simeq \frac{4}{\pi^2} \log \{\mu'(\omega)/\omega\}.$$

21. The (S^*, α) and S_{1-r} means

The (S^*, α) means, first introduced by Ramanujan*, are a form of quasi-Hausdorff method defined by the transform

$$(21.1) \qquad s_n^*(t) = \sum \binom{n+\nu}{\nu} s_\nu(t) \int_0^1 x^{n+1}(1-x)^\nu \, d\alpha(x),$$

where the weight function $\alpha(x)$ is of bounded variation in $0 \leq x \leq 1$. This transform is regular if and only if $\alpha(1) - \alpha(0+) = 1$ and $\alpha(1) = \alpha(1-)$. These means reduce to the S_{1-r} means defined by the transform

$$(21.2) \qquad \lambda_n(t) = r^{n+1} \sum s_\nu(t)(1-r)^\nu \binom{\nu}{n}$$

when $\alpha(x)$ is taken to be the function defined by (10.7). For regularity, $0 < r < 1$.

The following two theorems relating to the S_{1-r} means are due to Ishiguro [45, 1962]:

1. For the S_{1-r} means $\lambda_n(t)$, $0 < r < 1$, of the series $\psi(t)$, we have

$$(21.3) \qquad \lambda_n(t_n) \to \int_0^{\tau s} \frac{\sin \mu}{\mu} \, d\mu, \qquad n \to \infty,$$

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* Ramanujan, M. S., "On Hausdorff and quasi-Hausdorff methods of summability." Quart. Jour. Math. 8 (1957), 197-213.

whenever $nt_n \to \tau$, $0 \leq \tau \leq \infty$, and $nt_n^2 \to 0$ as $t_n \to 0$. Here s denotes the quantity $(1 - r)/r$.

2. For the S_{1-r} method, the Lebesgue constants are given by

$$(21.4) \qquad L_n(S_{1-r}) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left| \sum \binom{n+\nu}{\nu} r^{n+1}(1 - r)^{\nu} \frac{\sin(\nu+1/2)\mu}{2 \sin 1/2\mu} \right| d\mu$$

$$= \frac{2}{\pi^2} \log \frac{n}{1 - r} + a_o + O(1), \qquad n \to \infty,$$

where a_o is defined by (4.8) or (4.17).

The method of proof was essentially that of Szász [103] for the first theorem and that of Livingston [63] for the second theorem.

For Ramanujan's (S^*, α) means, Ishiguro [46, 1964] generalized the above result relating to Gibbs phenomenon in the following theorem :

For the regular (S^*, α) means $s_n^*(t)$ of $\psi(t)$, we have

$$(21.5) \qquad \lim s_n^*(t_n) = \int_o^1 \int_o^{\tau s} \frac{\sin \mu}{\mu}\, d\mu\, d\alpha(x) \qquad n \to \infty,$$

provided that the weight function $\alpha(x)$ is continuous at $x = 0$, $nt_n \to \tau$ and $nt_n^2 \to 0$. Here s has the same meaning as in Theorem 1.

From (21.5), it is easily seen that the Gibbs phenomenon is preserved completely by the S_{1-r} means, and (21.4) shows that the Lebesgue constants are unbounded.

22. Sonnenschein means

Let $f(z)$ be analytic in the neighbourhood of the origin and let

$$(22.1) \qquad \{f(z)\}^n = \sum f_{n\nu} z^\nu \qquad n = 1, 2, \ldots$$

$$f_{oo} = 1, \; f_{o\nu} = 0 \qquad \nu = 1, 2, \ldots$$

Set $F = (f_{n\nu})$ where $(f_{n\nu})$ is the matrix with the elements $f_{n\nu}$. This matrix is said to be generated by $f(z)$ and determines a sequence to sequence transformation. For a sequence $\{s_n(t)\}$, we have the transform

$$(22.2) \qquad \sigma_n(t) = \sum f_{n\nu} \, s_\nu(t) \, .$$

If $f(z)$ is analytic for $|z| < R$ where $1 < R$, and if $f(1) = 1$ and $|f(z)| < 1$ when $|z| \leq 1$, $z \neq 1$, the matrix generated by $f(z)$ is said to be a Sonnenschein matrix. The transformation is regular if $\operatorname{Re}A \neq 0$, where A is defined by

$$(22.3) \qquad f(z) - z^\zeta = Ai^p(z - 1)^p + O(1)(z - 1)^{p+1}, \quad z \to 1 \, ,$$

where $\zeta = f'(1) > 0$, and p is an even, positive integer.

For the series $\sum (2\nu - 1)^{-1} \sin(2\nu - 1)$, Sledd [93, 1961] proved that for the regular Sonnenschein transform,

$$(22.4) \qquad \sigma_n(t_n) = \int_0^\tau \frac{\sin \mu}{\mu} \, d\mu + O(n^{-1})$$

where $-\pi \leq \tau \leq \pi$. $nt_n^{\,2} \to 0$ and $nt_n \to \tau(2\zeta)^{-1}$. Thus Gibbs phenomenon is seen to be preserved completely. Here ζ denotes $f'(1)$, where $f(z)$ is the function generating the Sonnenschein matrix (F).

Sledd also proved that the Lebesgue constants are given by

$$(22.5) \qquad L_n(F) = \frac{4}{p\pi^2} \log \frac{n^{p-1}\zeta^p}{\xi} - \frac{4C}{p\pi^2} - \frac{4}{\pi^2} \log \frac{\pi}{2}$$

$$+ 2\int_o^1 \log \Gamma(t) \cos \pi t dt + o(1),$$

where C is the Euler-Macheroni constant, $\xi = -\mathrm{Re}A$, A and p being defined by (22.3). Since p is an even, positive integer, $\zeta > 0$ and $\xi > 0$, it is seen that the Lebesgue constants are unbounded.

23. <u>Trigonometric means</u>

Consider the Fourier sine series of a function, $f(t) = \sum b_\nu \sin \nu t$, and the partial sums $s_n(t) = \sum_1^n b_\nu \sin \nu t = \sum \nu b_\nu \frac{\sin \nu t}{\nu}$. This may be considered as a sine series or as a trigonometric transform of the sequence $\{\nu b_\nu\}$:

$$(23.1) \qquad \sigma_n(t) = \sum_1^n a_{n\nu}\, s_\nu$$

where $a_{n\nu} = \nu^{-1} \sin \nu t$ and $s_\nu = \nu b_\nu$. More generally, consider the transform

$$(23.2) \qquad \sigma_n(r_n, t_n) = \sum_1^n s_\nu r_n^\nu \nu^{-1} \sin \nu t_n, \quad r_n \to 1, \quad t_n \to 0.$$

For $s_\nu = \nu b_\nu$, this reduces to

$$(23.3) \qquad s_n(r_n, t_n) = \sum_1^n r_n^\nu b_\nu \sin \nu t_n$$

where $s_n(r, t)$ is the n^{th} partial sum of the harmonic series $\sum r^\nu b_\nu \sin \nu t$.

Szász [100, 101, 1943; 102, 1944] made a fairly detailed study of trigonometric transforms, in the course of which he proved the following theorem relating to Gibbs phenomenon:

Let $f(t) \sim \sum b_\nu \sin \nu t$ be an odd, periodic function of period 2π. Let $j = \lim_{t \to 0+} \frac{2}{t} \int_0^t f(\mu)\, d\mu$. If $\int_0^t |2f(\mu) - j|\, d\mu = O(t)$, $t \to 0^+$, then

$$s_n(t_n) \to \frac{j}{\pi} \int_0^\pi \frac{\sin \mu}{\mu}\, d\mu$$

whenever $nt_n - \pi = O(n^{-1})$ as $n \to \infty$. Here $s_n(t)$ denotes the n^{th} partial sum of the Fourier series of $f(t)$.

For other results relating to trigonometric means, the reader is referred to the original articles by Szász.

24. de la Vallee-Poussin sums

The de la Vallee-Poussin sums are defined by the series to sequence transform

$$(24.1) \qquad \sigma_n = \sum_1^n \frac{n!\, n!}{(n - \nu)!\,(n + \nu)!}\, a_\nu$$

or by the sequence to sequence transform

$$(24.2) \qquad \sigma_n = \sum_1^n \frac{n!\, n!\, (2\nu + 1)}{(n + \nu + 1)!\,(n - \nu)!}\, s_\nu$$

where s_ν are the partial sums of the series $\sum a_\nu$. The method is regular. In each case, the generalized sum is defined as the limit, if it exists, of the right hand side as n tends to infinity.

For a summability method defined by the transform

$$(24.3) \qquad \sigma_{m,n}(t) = (n+1)^{-1} \sum_{m-n}^{m} s_\nu(t),$$

Stečkin [96, 1951] proved that the Lebesgue constants $N_{m,n}$ are given by

$$(24.4) \qquad N_{m,n} = \frac{4}{\pi^2}\{\log \frac{m+1}{n+1} + 2\sum \frac{\log \nu}{4\nu^2-1} + 3 \log 2 + \gamma\} + O(\frac{n+1}{m+1}),$$

generalizing the result for ordinary convergence $(n = 0)$. He proved that if $m_p \geq n_p \to \infty$ over the even integers, and $r_p = \{2(m_p + 1)/(n_p + 1)\} - 1 \to r < \infty$, then

$$(24.5) \qquad N_{m_p n_p} \to \frac{2}{\pi} \int_0^\infty |\sin r\mu \, . \, \sin \mu| \frac{d\mu}{\mu^2} .$$

As a function of $(m + 1)/(n + 1)$, $N_{m,n}$ is shown to be increasing and concave.

CHAPTER III

DISCONTINUITIES OF THE SECOND KIND AND GENERALIZED FOURIER SERIES

In Chapter II we observed the behaviour of Gibbs phenomenon of the partial sums of ordinary Fourier series at jump discontinuities for various summability methods. In this chapter we will examine some results of a more general nature pertaining to discontinuities of the second kind, and some results pertaining to generalized Fourier series. We will observe that the (R, k), the $(C, 1)$, the $(C, 2)$ and the $(H, 2)$ means of generalized Fourier series present the Gibbs phenomenon at points of discontinuity of the associated function, although they do not exhibit it in the case of ordinary Fourier series.

Perhaps the earliest investigations of Gibbs phenomenon at points of discontinuity of the second kind were carried out by Kuttner [55, 56, 57, 58]. More recently Walmsley [115] and Lee [61], amongst others, have made important contributions. The literature in this field, however, is not nearly as extensive as it is in the field of ordinary Fourier series.

1. Some General Results

It is known that the validity of the proposition that the means of a given kind of the Fourier series of a positive function are all everywhere positive depends on the behaviour of the corresponding means of the formal series

$$D(t) = 1/2 + \sum \cos \nu t. \tag{1.1}$$

The n^{th} partial sum of this series,

$$(1.2) \qquad D_n(t) = 1/2 + \sum_1^n \cos \nu t = \frac{\sin(n + 1/2)t}{2 \sin 1/2\, t}$$

is the Dirichlet kernel. Its transform under a given summability method is generally denoted by $K_n(t)$. We give some results pertaining to the behaviour of Gibbs phenomenon obtained by an examination of the behaviour of the series (1.1) under a given summability method.

Theorem 1 (Kuttner [55, 1944]):

> If $0 < \lambda < 2$, there is a finite function $k(\lambda)$ such that the means (R, n^{λ}, κ) of the series (1.1) are everywhere positive if $\kappa \geq k(\lambda)$, but not for $\kappa < k(\lambda)$. The function $k(\lambda)$ is strictly increasing, continuous in $0 < \lambda < 2$ and tends to infinity as $\lambda \to 2$. Further, $k(1) = 1$, and if we denote the limit of $k(\lambda)$ as $\lambda \to 0$ by $k(0)$, then $0 < k(0) < 1$. If $\lambda = 2$, the means (R, n^{λ}, κ) of the series (1.1) are not, for any κ, everywhere positive, but the Abel means (A, n^{λ}) are. If $\lambda > 2$, neither the Riesz (R, n^{λ}, κ) means nor the Abel (A, n^{λ}) means are all everywhere positive.

Theorem 2 (Kuttner [56, 1944])

If $0 < \lambda < 2$, there is a function $r(\lambda)$ such that the Gibbs phenomenon vanishes for the (R, n^λ, κ) means of the Fourier series of a function having a simple discontinuity if $\kappa \geq r(\lambda)$, but not if $\kappa < r(\lambda)$. The function $r(\lambda)$ is continuous and strictly increasing, and is, for all $\lambda < 2$, less than the function $k(\lambda)$. It tends to 0 as $\lambda \to 0$, equals Cramér's constant r_o when $\lambda = 1$, and tends to infinity as $\lambda \to 2$. If $\lambda = 2$, the Gibbs phenomenon persists for the (R, n^λ, κ) means however large κ may be. If $\lambda > 2$, it persists for the Abel (A, n^λ) means, and hence necessarily for the (R, n^λ, κ) means.

Theorem 3 (Kuttner [57, 1945])

In order that a given K-method should be such that for any Lebesgue integrable function $f(t)$ the Gibbs phenomenon should not occur, it is necessary and sufficient that the kernel $K_n(t)$ should be bounded below for all n, t.

Kuttner illustrated Theorem 3 by showing that the kernel $K_n(t)$ for the (C, r) means is bounded below if and only if $r \geq 1$. It follows that provided $r < 1$, we can find a function the (C, r) means of whose Fourier series will exhibit the Gibbs phenomenon. By considering r

such that $r_0 \leq r < 1$, where r_0 is Cramér's constant, it is seen that there exist functions, with discontinuities of the second kind, the (C, r) means of whose Fourier series exhibit the Gibbs phenomenon for $r > r_0$.

Kuttner's last contribution in this field was related to summability methods, which he called the K' methods, in which, in particular, the generalized sum of the series $\sum a_\nu$ is defined by

$$(1.3) \qquad \lim_{\mu \to \infty} \sum q(\tfrac{\nu}{\mu})\, a_\nu .$$

For such methods, he proved that if the kernel $K_n(t)$ is bounded below for all $\mu > 0$, and all t, then it is positive for all $\mu > 0$ and all t [58, 1947]. The Riesz (R, n^λ, κ) and the Abel (A, n^λ) methods are included in this group. By the second theorem above it is seen that the Gibbs phenomenon will vanish at points of simple discontinuity for the (R, n^λ, κ) means whenever $\kappa \geq r(\lambda)$, and by the first we observe that the kernel $K_n(t)$ is positive everywhere if and only if $\kappa \geq k(\lambda)$. It follows that if we consider functions having discontinuities of the second kind, we can always find a function the (R, n^λ, κ) means of whose Fourier series will exhibit the Gibbs phenomenon whenever $\kappa < k(\lambda)$.

Izumi and Satô [49, 1956] demonstrated that there exists a function which presents the Gibbs phenomenon at a point of discontinuity of the second kind, and another function which does not present the Gibbs phenomenon at a point of discontinuity of the second kind. Also, they proved

Theorem 4

Let $f(t) = a\,\psi(t - \xi) + g(t)$ $\quad (\psi(t) \sim \sum \nu^{-1} \sin \nu t)$

be such that

$$(1.4) \qquad \limsup g(t) = 0 \qquad \liminf g(t) \geq -a\pi \qquad t \to \xi+$$

$$\limsup g(t) \leq a\pi \qquad \liminf g(t) = 0 \qquad t \to \xi-$$

$$(1.5) \qquad \int_o^t |g(\xi + \mu)|\, d\mu = o(|t|) .$$

Then the Gibbs phenomenon of $f(t)$ appears at $t = \xi$, and the Gibbs set contains the interval $[a(h + 1)\pi/4, - a(h + 1)\pi/4]$, where $h = 2/\pi \int_o^\pi \frac{\sin \mu}{\mu}\, d\mu$.

They proved that this theorem can be generalized by replacing (1.5) by the conditions

$$(1.6) \qquad \int_o^t g(\xi + \mu) d\mu = o(|t|)$$

$$\int_o^t \{g(x + \mu) - g(x - \mu)\} d\mu = o(|t|) .$$

Ishiguro [39, 1956] proved that the (C, r) means of the Fourier series of the class of functions satisfying the hypothesis of Theorem 4, above, will show the Gibbs phenomenon for $r < r_o$, but not for $r \geq r_o$, where r_o is Cramér's constant. In a subsequent paper [41, 1957], he proved that this result also holds for the more general class of functions satisfying the hypothesis of Theorem 4 with condition (1.5) replaced by condition (1.6).

Izumi and Satô [50, 1957] continued work in the field and produced a number of additional results. Among these, they proved

Theorem 5

If $\int_0^t \{f(x + \mu) - f(x - \mu)\}d\mu = o(-t/\log t)$, uniformly in x, then the partial sums of the Fourier series of $f(t)$ do not present the Gibbs phenomenon;

Theorem 6

If $\int_0^t \{f(x + \mu) - f(x - \mu)\}d\mu = o(t)$, uniformly in x, then the Cesáro means of the Fourier series of $f(t)$ of positive order do not present the Gibbs phenomenon.

In addition, they gave the construction of a function $f(t)$ such that the partial sums of its Fourier series present the Gibbs phenomenon, but its Cesáro means of any positive order do not. The function which they constructed is the following:

Let $n_k = 2^{2^k}$, and let $\varphi_k(t)$ be an even, concave function which is zero for $t \geq \pi/2n_k$, and touches the y-axis at $y = 1$ and the t-axis at $t = \pi/2n_k$. Define

$$f_k(t) = \varphi_k\{t-(2j-1/2)\pi/n_k\} \quad \text{in} \quad \{(2j-1)\pi/n_k,\ 2j\,\pi/n_k\}$$

$$= 0 \quad \text{elsewhere}$$

$$f(t) = \sum f_\nu(t)$$

where

$$j = 2^{2^k-(k+1)},\ 2^{2^k-(k+1)}+1,\ 2^{2^k-(k+1)}+2,\ \dots,$$

$$2^{2^k-1}\ .$$

(In the published account the functions $f_k(t)$ were defined by

$$f_k(t) = \varphi_k\{t + (2j-1/2)\pi/n_k\} \quad \text{in} \quad \{(2j-1)\pi/n_k,\ 2j\,\pi/n_k\}$$

$$= 0 \quad \text{elsewhere.}$$

This must clearly be a typographic error reducing $f_k(t)$, and so $f(t)$, identically to zero according to the definition of $\varphi_k(t)$).

Ishiguro [42, 1960] states that Izumi and Satô [49] proved the following

Theorem 7

There exists a function $f(t)$ which satisfies

$$\int_0^t f(\mu)d\mu = o(|t|)$$

$$\int_0^t \{f(x+\mu) - f(x-\mu)\}d\mu = o(|t|) \text{ uniformly in } x$$

and presents the Gibbs phenomenon for classic convergence.

Ishiguro then sets out to prove that a function satisfying Theorem 7 does not exhibit the Gibbs phenomenon for the Borel, Euler and Hausdorff means.

The result that Ishiguro is referring to could be Theorem 4 with condition (1.5) replaced by (1.6) since the reference that Ishiguro gives does not appear to deal with Theorem 1.7. If that should be the case, however, then if we remove some of the lack of precision in the wording of Theorem 4 by specifying that $a \neq 0$, which is what is implied, then Theorems 4 and 7 are distinctly different.

2. Generalized Fourier Series

We have seen that to examine the behaviour of the Gibbs phenomenon at a point of simple discontinuity in the case of functions of bounded variation, it is sufficient to examine its behaviour for a specific such function. Now if a function $f(t)$ has a pole at its point of discontinuity, we can express it in the form

$$f(t) = k \, . \, \chi(t) + g(t) \tag{2.1}$$

where $\chi(t)$ is some specific such function with a pole of the same order at the point of discontinuity of $f(t)$, and $g(t)$ is continuous. Hence by analogy, to study the behaviour of the Gibbs phenomenon at a pole of a function, it is sufficient to study it for a specific such function. There is, however, this difference from the case of functions of bounded variation with simple discontinuities. In the case of functions which have a pole at a point of discontinuity, it is no longer meaningful to consider the Gibbs set, so that we are led to the examination of the Gibbs pseudo ratio.

For the case of functions with a pole of order one, Walmsley [115, 1953] studied the function $f(t) = 1/2 \cot t/2$. To expand this in a Fourier series, we have

$$\text{(2.2)} \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt; \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt.$$

The first of these integrals does not exist in the usual sense because the integral has a pole at $t = 0$. However, interpreted as a Cauchy principal value integral $\left\{ \lim_{\epsilon \to 0} \left(\int_{-\pi}^{-\epsilon} + \int_{\epsilon}^{\pi} \right) \right\}$, its value is zero for all n. The second integral does exist in the usual sense and yields, for all n, $b_n = 1$. The resulting Cauchy principal value Fourier series is given by

$$\text{(2.3)} \qquad 1/2 \cot t/2 \quad \sim \quad \sum \sin \nu t.$$

For each value of t, the series on the right is summable $(C, 1)$ to sum $1/2 \cot t/2$. Denoting the $(C, 1)$ transform of the series on the right in (2.3) by $\sigma_n^{(1)}(t)$, and the partial sums by $s_n(t)$, Walmsley obtained

$$\text{(2.4)} \qquad s_n(t) = 1/2 \cot t/2 - 1/2 \operatorname{cosec} t/2 \cos(n + 1/2)t$$

$$\text{(2.5)} \qquad \sigma_n^{(1)}(t) = n^{-1} \sum_1^n s_\nu(t)$$

$$= 1/2 \cot t/2 + 1/2n \cot t/2$$

$$- 1/4n \operatorname{cosec}^2 t/2 \sin(n + 1)t.$$

To determine the behaviour of the Gibbs phenomenon at $t = 0$ for the $(C, 1)$ means, he set

$$\text{(2.6)} \qquad \rho_n^{(1)} = \sigma_n^1(t)/\{1/2 \cot t/2\}$$

$$= 1 + 1/n - 1/n \operatorname{cosec} t \sin(n + 1)t.$$

For $t = t_n$ and $nt_n \to \tau$, $0 \leq \tau \leq \infty$ $n \to \infty$, he obtained

$$(2.7) \qquad \rho_n^{(1)} = 1 + 1/n - \sin(nt_n + t_n) \cdot \frac{t_n}{\sin t_n} \cdot \frac{1}{nt_n}$$

and in the limit,

$$(2.8) \qquad \rho^{(1)} = 1 - \frac{\sin \tau}{\tau}, \qquad n \to \infty.$$

As τ goes from 0 to 4.4934... to 7.7253... , $\rho^{(1)}$ goes from 0 to a maximum of 1.2172... to a minimum of 0.8716... , tending to a limiting value of unity at $\tau = \infty$. Thus the (C, 1) means of the Cauchy principal value Fourier series of $1/2 \cot t/2$ are seen to exhibit a Gibbs phenomenon at $t = 0$ quite similar to that of functions which have a simple discontinuity, with a Gibbs pseudo ratio, PR, of 1.2172... .

For the case of a function with a pole of order 2 at $t = 0$, Walmsley (ibid) considered the function $f(t) = -1/4 \operatorname{cosec}^2 t/2$. Since in this case the integrals in (2.2) do not exist in any real sense, they may be evaluated in the complex sense along any path which avoids the origin. In this manner, for any path Γ from $-\pi$ to π, avoiding the origin,

$$(2.9) \qquad a_o = 1/\pi \int_\Gamma - 1/4 \operatorname{cosec}^2 t/2 \, dt = 0$$

$$(2.10) \qquad a_n = 1/\pi \int_\Gamma \cos nt \, d(1/2 \cot t/2)$$

$$= n/\pi \int_\Gamma 1/2 \cot t/2 \sin nt dt \qquad \text{by parts}$$

$$= n/\pi \int_{-\pi}^{\pi} 1/2 \cot t/2 \sin nt dt$$

$$= n$$

as a Cauchy principal value integral. Similarly

$$(2.11) \qquad b_n = 1/\pi \int_{\Gamma} - 1/4 \operatorname{cosec}^2 t/2 \sin nt dt$$

$$= \mp\ i\ .\ \{\text{res. of } - 1/4 \operatorname{cosec}^2 t/2 \sin nt \text{ at } t = 0\}$$

$$= \pm\ in$$

where $b_n = +\ in$ if the path of integration is above the origin. Thus, the generalized Fourier series corresponding to Γ are

$$(2.12) \qquad -1/4 \operatorname{cosec}^2 t/2 \sim \sum (\nu \cos \nu t \pm i\nu \sin \nu t).$$

Suppressing the imaginary part, he obtained the standard generalized Fourier series

$$(2.13) \qquad - 1/4 \operatorname{cosec}^2 t/2 \sim \sum \nu \cos \nu t.$$

Proceeding as in the case of the series (2.3), Walmsley obtained the (C, 2) transform $\sigma_n^{(2)}(t)$ of (2.13) in closed form

$$(2.14) \qquad \sigma_n^{(2)}(t) = \operatorname{cosec}^2 t/2\{- \frac{n+3}{4(n+1)} + \frac{3}{8n(n+1)} (\cos 3t/2 - \cos(n + 3/2)t)\} + \frac{1}{4n(n+1)} \operatorname{cosec}^3 t/2 \{3/2 \sin 3t/2 - (n + 3/2)\sin(n + 3/2)t\}$$

and

$$(2.15) \quad \rho_n^{(2)} = \{\sigma_n^{(2)}(t)\}/\{-1/4 \operatorname{cosec}^2 t/2\}$$

$$= \frac{n+3}{n+1} - \frac{3}{2n(n+1)} \operatorname{cosec}^2 t/2 \cos t/2\{\cos 3t/2 - \cos(n+3/2)t$$

$$- \frac{3}{2n(n+1)} \operatorname{cosec} t/2 \sin 3t/2 + \frac{2n+3}{2n(n+1)} \operatorname{cosec} t/2$$

$$. \sin(n+3/2)t.$$

Letting nt_n approach τ, $0 \leq \tau \leq \infty$, as n tends to infinity, he obtained in the limit,

$$(2.16) \quad \rho^{(2)} = 1 + \frac{2 \sin \tau}{\tau} - \frac{6(1 - \cos \tau)}{\tau^2} .$$

As τ goes from 0 to 2.6062... to 7.4145..., $\rho^{(2)}$ goes from 0 to -0.2516... to 1.1814... , and thereafter has alternate minima and maxima tending to unity as $\tau \to \infty$. The Gibbs phenomenon is seen to be present and the Gibbs pseudo ratio is $\{1.1814... + 0.2516...\} \div 1 =$ 2.4330.

For the Hölder (H, 2) transform of the series (2.13), Walmsley obtained

$$(2.17) \quad H_n^{(2)}(t) = -1/4 \operatorname{cosec}^2 t/2\{1 + \frac{1}{n} \sum_1^n \nu^{-1} +$$

$$\frac{1}{n} \sum_1^n (1 + \nu^{-1}) \cos(\nu + 1)t -$$

$$\frac{1}{n} \cot t/2 \sum_1^n \nu^{-1} \sin(\nu + 1)t\}.$$

Setting $\rho_n^{(2)}(H) = H_n^{(2)}(t)/\{-1/4 \operatorname{cosec}^2 t/2\}$, $nt_n \to \tau$, $0 \leq \tau \leq \infty$, and letting $n \to \infty$, he obtained

$$(2.18) \qquad \rho^{(2)}(H) = 1 + \frac{\sin \tau}{\tau} - \frac{2 \,\mathrm{Si}\, \tau}{\tau}$$

where $\mathrm{Si}\, \tau = \int_0^{\tau} \frac{\sin \mu}{\mu} \, d\mu$.

Again as τ increases from 0 to 2.7820... to 7.8687... , $\rho^{(2)}(H)$ goes from 0 to -0.1890... to 0.7311... , and thereafer has alternate minima and maxima all tending to unity from below. Thus, the (H, 2) means of the series (2.13) show a one sides Gibbs phenomenon with a Gibbs pseudo ratio of $\{1 + 0.1890...\} + 1 = 2.1890...$.

Lee [61, 1959] extended Walmsley's investigations to the (R, k) means of the series (2.3), and also to the generalized Fourier series

$$(2.19) \quad (a) \quad (1/2 \cot t/2)^{(4m+1)} \sim \sum \nu^{4m+1} \cos \nu t$$

$$(b) \quad (1/2 \cot t/2)^{(4m-1)} \sim \sum -\nu^{4m-1} \cos \nu t$$

$$(c) \quad (1/2 \cot t/2)^{(4m)} \sim \sum \nu^{4m} \sin \nu t$$

$$(d) \quad (1/2 \cot t/2)^{(4m-2)} \sim \sum -\nu^{4m-2} \sin \nu t$$

where the exponent on the left indicates repeated differentiation and $m = 1, 2, \ldots$. Denoting the (R, k) transform by $R_h^k(t)$, Lee obtained the following results:

For the Cauchy principal value Fourier series (2.3), the set of all limiting values of the ratio $\{R_h^1(t)\}/\{1/2 \cot t/2\}$

is the infinite interval $[0, \infty]$ as $t \to 0+$, $h \to 0$, $t/h \to \tau$, and $0 \leq \tau$, $\tau \neq 1$.

For the real generalized Fourier series (2.19(a)), $(m = 0)$, the set of all limiting values of the ratio $\{R_h^2(t)\}/\{1/2 \cot t/2\}$ is the infinite interval $[0, \infty]$ at $t \to 0+$, $h \to 0$, $t/h \to \tau$, and $0 \leq \tau$, $\tau \neq 2$,

and in general, for the generalized Fourier series (2.19), $m \neq 0$,

The set of all limiting values of the ratio $\{R_h^k(t)\}/\{1/2 \cot t/2\}^{(k-1)}$ is the whole real number axis $[-\infty, \infty]$ at $t \to 0+$, $t/h \to \tau$ and $0 \leq \tau$ (some integers excluded).

From these results, it follows that the (R, k) means of the generalized Fourier series present the Gibbs phenomenon at points of discontinuity, although they do not exhibit it in the case of ordinary Fourier series (Lee [60, 1956]; Chapter II, δ 18).

CHAPTER IV

SOME RESULTS PERTAINING TO MULTIPLE FOURIER SERIES

1. Gibbs Phenomenon

Cheng [11] defined the Gibbs phenomenon for double Fourier series in a manner analagous to the following:

Let $\{f_n(x, y)\}$ be a sequence of real valued functions defined in a neighbourhood of a point (x_o, y_o) in the Euclidean plane. Let $\{f_n(x, y)\}$ converge to a function $f(x, y)$ for $x_o < x \leq x_o + h$ and $y_o < y \leq y_o + k$, and let $f(x_o+, y_o+)$ exist. The sequence $\{f_n(x, y)\}$ is said to present the Gibbs phenomenon in the UR neighbourhood of (x_o, y_o) if one or both of the following are true:

$$(1.1) \qquad \limsup_{\substack{(x,y) \to (x_o,y_o) \\ n \to \infty}} f_n(x, y) > f(x_o+, y_o+)$$

$$(1.2) \qquad \liminf_{\substack{(x,y) \to (x_o,y_o) \\ n \to \infty}} f_n(x, y) < f(x_o+, y_o+) .$$

The Gibbs phenomenon in the UL, LL and LR neighbourhoods is similarly defined. The sequence $\{f_n(x, y)\}$ is then said to exhibit the Gibbs phenomenon at a point (x_o, y_o) if it exhibits the Gibbs phenomenon in at least one of the four heighbourhoods of (x_o, y_o).

By considering, for example, the sequence $\{\arctan n(x + y)\}$, it is seen that the above definition has the same weakness as Zygmund's

definition of the Gibbs phenomenon for ordinary series, according to which if a sequence of continuous functions fails to present the Gibbs phenomenon at a given point, then the function to which the sequence converges must necessarily be continuous at that point. See Zygmund [127] and Forbes [22]. This weakness may be removed by defining the Gibbs phenomenon for double Fourier series in the following manner:

Let $f_n(x, y)$ be a sequence of real-valued functions converging to a limit function $f(x, y)$. The sequence $\{f_n(x, y)\}$ is said to exhibit the Gibbs phenomenon at (x_o, y_o) if either or both of the following are true:

$$(1.3) \qquad \limsup_{\substack{(x,y) \to (x_o,y_o) \\ n \to \infty}} f_n(x, y) > \limsup_{(x,y) \to (x_o,y_o)} f(x, y)$$

$$(1.4) \qquad \liminf_{\substack{(x,y) \to (x_o,y_o) \\ n \to \infty}} f_n(x, y) < \liminf_{(x,y) \quad (x_o,y_o)} f(x, y) .$$

The definition of Gibbs phenomenon for double series allows an easy extension to its definition for a sequence of functions in n variables.

Now let $f(x, y)$ be a real valued, periodic function with period 2π in each variable, and let $f(x, y)$ be Lebesgue integrable. Its Fourier series may be written in the form

$$(1.5) \qquad f(x, y) \sim \sum_{-\infty}^{\infty} c_{mn} e^{i(mx+ny)}$$

where

$$(1.6) \qquad c_{mn} = (2\pi)^{-2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(r, s) e^{-i(mr + ns)} dr ds.$$

The R^{th} Riesz circular mean of order κ of (1.5) is given by

$$(1.7) \qquad \sum_{\nu^2 \leq R^2} \left(1 - \frac{\nu^2}{R^2}\right)^{\kappa} \left(\sum_{m^2+n^2=\nu^2} c_{mn} e^{i(mx + ny)} \right)$$

The extension of the definition of Fourier series to an arbitrary n-dimensional space is as follows:

$$(1.8) \qquad f(x) \sim \sum_{-\infty}^{\infty} a_{\nu} e^{i\nu . x}$$

where now $x = (x_1, x_2, \ldots, x_n)$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ and $\nu . x = \nu_1 x_1 + \nu_2 x_2 + \ldots + \nu_n x_n$. The coefficients a_ν are given by

$$(1.9) \qquad a_{\nu} = (2\pi)^{-n} \int_{Q_n} f(x) e^{-i\nu . x} dx$$

where Q_n is the fundamental cube $-\pi < x_i \leq \pi$, $i = 1, 2, \ldots, n$. The spherical Riesz means (R, n^2, κ) are defined by

$$(1.10) \qquad R_R^{\kappa}(x) = \sum_{|\nu| < R} \left(1 - \frac{|\nu|^2}{R^2}\right)^{\kappa} a_{\nu} e^{i\nu . x}$$

and the transform of the kernel by

$$(1.11) \qquad K_R^{\kappa}(x) = \sum_{|\nu| < R} \left(1 - \frac{|\nu|^2}{R^2}\right)^{\kappa} e^{i\nu . x} .$$

Cheng [11, 1950] considered the case of double Fourier series and proved the following theorem:

Let $f(x, y) = f_1(x) \cdot f_2(y)$, where $f_1(x)$ and $f_2(y)$ are of bounded variation and periodic with period 2π, let R_R^κ be the circular Riesz mean of the Fourier series of $f(x, y)$, and let (x_o, y_o) be a point of discontinuity of $f(x, y)$. Then the circular Riesz means present the Gibbs phenomenon in the neighbourhood of (x_o, y_o) for every $\kappa > 0$.

Cheng first proved the result for the UR neighbourhood of a point of discontinuity of the function $f(x, y) = h(x) \cdot h(y)$, where $h(x) \sim \sum \nu^{-1} \sin \nu x$, and then extended this to obtain the theorem.

2. Lebesgue Constants

Šerbina [91, 1948] defined the generalized Féjer means of the double Fourier series of a function $f(x, y)$ by

$$(2.1) \qquad \sigma_{mn}^{pq} = (p + 1)^{-1}(q + 1)^{-1} \sum_{\mu=m-p}^{m} \sum_{\nu=n-q}^{n} s_{\mu\nu} ,$$

where $p = p(m)$ $(0 \leq p \leq m)$ and $q = q(n)$ $(0 \leq q \leq n)$ are functions of m and n respectively. The conditions

$$(2.2) \qquad \liminf_{m \to \infty} p(m)/m = \alpha > 0, \quad \liminf_{n \to \infty} q(n)/n = \beta > 0$$

are necessary and sufficient for the uniform convergence of σ_{mn}^{pq} to $f(x, y)$. Šerbina obtained an estimate for the Lebesgue constants M_{mn}^{pq} for these means:

$$(2.3) \qquad M_{mn}^{pq} = \frac{16}{\pi^4} \log \frac{m}{p+1} \log \frac{n}{q+1} + O(\log \frac{m}{p+1}) + O(\log \frac{n}{q+1}) + O(1).$$

Thus, if conditions (2.2) are not satisfied, it is seen that the Lebesgue constants are unbounded.

The asymptotic estimates for the Lebesgue constants for the Riemann means of double Fourier series were obtained by Pavlov'skiǐ [84, 1962]. However, at the present time his results are not available.

Stein [99, 1961] studied Riesz's spherical means of the multiple Fourier series (1.8) and proved the following theorem pertaining to the Lebesgue constants:

> There exists a constant α, depending only on the number of dimensions n, so that the Lebesgue constants L_R for Riesz's spherical means (R, n^2, δ) for the critical exponent $\delta = 1/2(n - 1)$ are given by

$$(2.4) \qquad L_R = \alpha \log R + O(1).$$

(<u>Note</u>: If $\delta > 1/2(n - 1)$, the Riesz's spherical means of the Fourier series of a function $f(x)$ converge almost everywhere to $f(x)$, and the convergence is uniform if $f(x)$ is continuous. This convergence depends only on the values of $f(x)$ in any neighbourhood of x.

When δ is less than or equal to the critical exponent $1/2(n - 1)$, this is no longer generally true.)

We give here a brief outline of Stein's proof. Let $S_R^\delta(x)$ denote Riesz's spherical means of (1.8):

$$(2.5) \qquad S_R^\delta(x) = \sum_{|\nu| < R} \left(1 - \frac{|\nu|^2}{R^2}\right)^\delta a_\nu e^{i\nu . x}, \quad \delta = 1/2(n - 1).$$

Denote the kernel by

$$(2.6) \qquad D_R(x) = \sum_{|\nu| < R} \left(1 - \frac{|\nu|^2}{R^2}\right)^\delta e^{i\nu . x} .$$

For $n = 1$, (2.6) reduces to the Dirichlet kernel

$$(2.7) \qquad D_R(x) = \frac{\sin (N + 1/2)x}{\sin 1/2 \; x} , \qquad N = [R].$$

The Fourier integral analogue of (2.6) is given by

$$(2.8) \qquad H_R(x) = \int_{|\mu| < R} \left(1 - \frac{|\mu|^2}{R^2}\right)^\delta e^{i\mu . x} \, d\mu$$

which reduces to the Fourier integral analogue of (2.7) for $n = 1$:

$$(2.9) \qquad H_R(x) = \frac{2 \sin Rx}{x} .$$

Stein defined

$$(2.10) \qquad \Delta_R(x) = D_R(x) - H_R(x)$$

and proved that for $1 \leq q < \infty$,

$$(2.11) \qquad \sup_{0 \le R < \infty} \| \Delta_R(x) \|_q \le A_q \le qA$$

where the norm is the standard L^q norm taken over the fundamental cube, and A is some constant. He first obtained the result for $q = 2$ and then extended it to the general case for $q \ge 1$.

Taking $q = 1$ in (2.11), it follows that

$$(2.12) \qquad L_R = \int_Q |D_R(x)|\,dx$$

$$= \int_Q |H_R(x)|\,dx + O(1).$$

After proving that $H_R(x)$ is uniformly bounded for $|x| \ge \epsilon > 0$, the author showed that

$$(2.13) \qquad L_R = \int_{|x| \le \epsilon} |H_R(x)|\,dx + O(1).$$

Expressing $H_R(x)$ in terms of the Bessel functions,

$$(2.14) \qquad H_R(x) = 2^{\delta}\Gamma(\delta + 1)(2\pi)^{\delta+1/2} R^{1/2} J_{2\delta+1/2}(R|x|)|x|^{-(2\delta+1/2)}$$

and using an asymptotic estimate for the Bessel function, he showed further that

$$(2.15) \qquad L_R = \alpha_1 R^{1/2} \int_0^{\epsilon} |J_{2\delta+1/2}(R\mu)|\mu^{-1/2}d\mu$$

$$= \alpha_1 \int_0^{\epsilon R} |J_{2\delta+1/2}(\mu)|\mu^{-1/2}d\mu$$

$$= \alpha_1 \int_1^{\epsilon R} |J_{2\delta+1/2}(\mu)|\mu^{-1/2}d\mu + O(1)$$

$$= \alpha_2 \int_1^{\epsilon R} |\cos(\mu - (\delta + 1/2)\pi/2) \frac{d\mu}{\mu}$$

$$= \alpha_3 \log \epsilon R + O(1) \qquad R \to \infty$$

where the last result follows by integrating by parts. This completes the proof of his theorem.

From (2.15) it follows that there exists a continuous, periodic function whose Fourier expansion is not summable by Riesz's spherical means of the critical order δ. Stein also announced the result that there exists an integrable function whose Fourier series is almost everywhere non-summable by these means of the critical order. His earlier work [98], on the other hand, implies that if $|f| (\log^+ |f|)^2$* is integrable, then the Fourier series of $f(x)$ is almost everywhere (R, n^2, δ) summable.

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* $\log^+ |f| = \log |f|, \quad |f| \geq 1$

$\qquad = 0$ otherwise .

CHAPTER V

GIBBS PHENOMENON AND LEBESGUE CONSTANTS FOR OTHER ORTHONORMAL SYSTEMS AND SOME ADDITIONAL GENERAL RESULTS

In the course of logical development, the concepts of Gibbs phenomenon and Lebesgue constants were first considered for ordinary trigonometric Fourier series. The trigonometric system of functions, however, is but a part of a large class of orthonormal systems. We now extend these concepts to the more general case.

Let* $\{\varphi_n\}$ be an orthonormal set of functions on a space X. Suppose that a sequence of numbers $\{c_n\}$ is such that $\sum_n |c_n|^2$ converges. Let $s_n = \sum_1^n c_\nu\varphi_\nu$. Then there exists a square integrable function f such that $\{s_n\}$ converges to f, and

$$f \sim \sum c_\nu\varphi_\nu .$$

The coefficients c_ν are given by

$$c_\nu = \int_X f\varphi_\nu d\mu$$

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* This is essentially the Riesz-Fischer theorem. See Rudin, W., Principles of Mathematical Analysis. McGraw Hill Book Co., (1964), p. 255.

In general, given any linearly independent set of functions, $\{x_n\}$, over a space X, these functions may be transformed into an orthonormal set $\{\varphi_n\}$ by means of the Gram-Schmidt orthogonalization process. Thus, if we take $\{x_n\} = \{x^n\}$ on the interval $[-1, 1]$, the orthonormal set $\{\varphi_n\}$ is the set of normalized Legendre polynomials. If $\{x_n\} = \{x^n e^{-x^2/2}\}$, and the space is the entire real line, then the Gram-Schmidt orthogonalization process yields the set of normalized Hermite functions; and if the space is the segment $[0, \infty]$, and we take $\{x_n\} = \{x^n e^{-x}\}$, then we obtain the set of normalized Laguerre functions.

Thus it is seen that the ordinary Fourier series of functions may be described, and some of their properties studied, in this more general setting. It becomes natural, as an extension of the work on ordinary Fourier series, to inquire into the behavior of Gibbs phenomenon and Lebesgue constants in orthonormal systems other than the trigonometric system.

The definitions of Gibbs phenomenon and Lebesgue constants, as given in Chapter I, are sufficiently general so as to be immediately applicable here.

1. Generalizations of Integral Representations of Some Lebesgue Constants

Let $g(t)$ be Lebesgue integrable over $(0, \pi)$, of period π, and such that

$$(1.1) \qquad \int_0^1 \frac{g(t)}{t}\, dt$$

exists in some sense. Let

$$(1.2) \qquad L(x,\ b;\ g) = \frac{1}{b}\int_0^b \frac{g\{(2x+1)t\}}{\sin t}\, dt, \qquad 0 < b < \pi$$

$$(1.3) \qquad L_E(x,\ b;\ g) = \frac{1}{b}\int_0^b |\cos t|^x \frac{g\{(x+1)t\}}{\sin t}\, dt, \qquad 0 < b < \pi$$

$$(1.4) \qquad L_B^o(x,\ b;\ g) = \frac{1}{b}\int_0^b \exp(-2xt^2) \frac{g\{(2x+1)t\}}{t}\, dt \qquad 0 < b$$

$$(1.5) \qquad L_1(\xi,\ b;\ g;\ \tau) = \frac{1}{b}\int_0^{1/\tau(\xi)} \frac{g(\xi t)}{t}\, dt, \qquad 0 < \tau(\xi) = o(\xi)\ \xi \to \infty$$

Lorch [68, 1953] proved that

$$(1.2') \qquad L(x,\ b;\ g) = \frac{m}{b}\log x + \frac{2m}{b}\log 2 + \frac{m}{b}\log\tan(b/2) + \frac{1}{b}\int_0^1 \frac{g(t)}{t}\, dt - \frac{1}{b}\int_1^\infty \{m-g(t)\}\frac{dt}{t} + O(\tfrac{1}{x}), \quad x \to \infty.$$

If, in addition, $g(t)$ is even, and the series conjugate to the Fourier series of $g(t)$ is also a Fourier series, $b = \pi/2$ and x is taken as an integer n, then the error term $O(1/x)$ equals

$$(1.2'(a)) \qquad m/(\pi n) + o(1/n).$$

$$(1.3') \qquad L_E(x,\ b;\ g) = \frac{m}{2b}\log x - \frac{m}{2b}C + \frac{m}{2b}\log 2 + \frac{1}{b}\int_0^1 \frac{g(t)}{t}\, dt - \frac{1}{b}\int_1^\infty \{m-g(t)\}\frac{dt}{t} + O(1/x^{\frac{1}{2}}),\ x \to \infty$$

where C is the Euler-Mascheroni constant.

$$(1.4') \quad L_B^o(x, b; g) = \frac{m}{2b} \log x - \frac{m}{2b} C + \frac{m}{2b} \log 2 + \frac{1}{b}\int_o^1 \frac{g(t)}{t}\, dt$$

$$- \frac{1}{b}\int_1^\infty \{m-g(t)\} \frac{dt}{t} + O(1/x^{\frac{1}{2}}), \qquad x \to \infty$$

$$(1.5') \quad L_1(\xi, b; g; \tau) = \frac{m}{b} \log \{\xi/\tau(\xi)\} + \frac{1}{b}\int_o^1 \frac{g(t)}{t}\, dt$$

$$- \frac{1}{b}\int_1^\infty \{m-g(t)\} \frac{dt}{t} + O(\tau(\xi)/\xi$$

If, in addition, $g(t)$ is symmetric about δ, where $\delta = 0$ or $\delta = \pi/2$, and if $\xi/\tau(\xi)$ becomes infinite through the subsequence $\{n\ \pi/2 + \delta\}$, then $O(\tau(\xi)/\xi)$ can be replaced by $O(1/n^2)$.

In all the above cases, m denotes the mean value of $g(t)$ over a complete period.

Specializing the above results, if $g(t) = |\sin t|$, $b = \pi/2$ and $m = \frac{1}{\pi}\int_o^\pi \sin t\ dt = 2/\pi$, then

$$(1.2'') \quad L(n, \pi/2; g) = \frac{4}{\pi^2} \log n + \frac{8}{\pi^2} \log 2 + \frac{2}{\pi}\int_o^1 \frac{\sin t}{t}\, dt$$

$$- \frac{2}{\pi}\int_1^\infty \{\frac{2}{\pi} - |\sin t|\} \frac{dt}{t} + O(1/n), \qquad n \to \infty$$

$$(1.3'') \quad L_E(n, \pi/2; g) = \frac{2}{\pi^2} \log n - \frac{2}{\pi^2} C + \frac{2}{\pi^2} \log 2 + \frac{2}{\pi}\int_o^1 \frac{\sin t}{t}\, dt$$

$$- \frac{2}{\pi}\int_1^\infty \{\frac{2}{\pi} - |\sin t|\} \frac{dt}{t} + O(1/n^{\frac{1}{2}}), \qquad n \to \infty.$$

$$= \frac{2}{\pi^2} \log n + a_o + O(1/n^{\frac{1}{2}}).$$

$$(1.4'') \quad L_B^o(x, \pi/2; g) = \frac{2}{\pi^2}\log x - \frac{2}{\pi^2} C + \frac{2}{\pi^2}\log 2 + \frac{2}{\pi}\int_o^1 \frac{\sin t}{t}\, dt$$

$$- \frac{2}{\pi}\int_1^\infty \{\frac{2}{\pi} - |\sin t|\}\frac{dt}{t} + O(1/x^{\frac{1}{2}}), \qquad x \to \infty$$

$$= \frac{2}{\pi^2}\log n + a_o + O(1/n^{\frac{1}{2}}), \quad \text{for} \quad x = n,$$

where a_o is the constant defined by (4.17, Chapter II). If, additionally, $\tau(\xi) = \xi^{\frac{1}{2}}$, then

$$(1.5'') \quad L_1(\xi, \pi/2; g; \tau) = \frac{2}{\pi^2}\log \xi + \frac{2}{\pi}\int_o^1 \frac{\sin t}{t}\, dt$$

$$- \frac{2}{\pi}\int_1^\infty \{\frac{2}{\pi} - |\sin t|\}\frac{dt}{t} + O(1/\xi^{\frac{1}{2}}), \quad \xi \to \infty .$$

(1.2") is recognized as the asymptotic expression for the Lebesgue constants for ordinary convergence of Fourier series when (1.2'(a)) is taken into account. (1.3") is the corresponding expression for the $(\varepsilon, \frac{1}{2})$ means, and, except for the additive error $O(1/x^{\frac{1}{2}})$, (1.4") is the asymptotic expression for the Lebesgue constants for the Borel means. As Lorch pointed out, (1.5") is the asymptotic expression for the Lebesgue constants for Hardy's* equivalent of Borel's method of summing Fourier series of continuous functions.

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* Hardy, G. H., Remarks on some points in the theory of divergent series. Annals of Math., (2), 36 (1935), 167-182.

2. The Prime Number Theorem

For the summation over the primes p, the series

$$(2.1) \qquad \sum p^{-1} \sin(t \log p),$$

being the imaginary part of $\sum p^{-(1+it)}$, is uniformly convergent for $\epsilon < t < T$, where $0 < \epsilon$, and convergent for all t. According to Mertens*, this is substantially equivalent to the non-vanishing of the Riemann ζ-function on the line $\sigma = 1$, and therefore to the prime number theorem. Due to the saltus of the argument $\arg \zeta(1 + it)$ at $t = 0$, where $\zeta(s) \sim (s-1)^{-1}$ and $s = 1 + it$, the series (2.1) is not uniformly convergent near $t = 0$.

Wintner [124, 1945] proved the following theorem:

The partial sums of the series (2.1) exhibit a Gibbs phenomenon at $t = 0$, in the sense that, as $n \to \infty$, the difference

$$(2.2) \qquad \sum_{p \leq n} p^{-1} \sin(t \log p) - \int_0^{t \log n} \mu^{-1} \sin \mu \, du$$

tends to a finite limit uniformly for $|t| < T$, where T is arbitrary but fixed.

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* Mertens, F., Über die Konvergenz einer aus Primzahlpotenzen gebildeten unendlichen Reihe. Gottinger Nachrichten (1887), 265-269.

To prove this result, Wintner first proved that it is sufficient to prove the uniform convergence of the Dirichlet series

$$(2.3) \quad \sum a_\nu \nu^{-s} = \sum p^{-s} - \sum_2^\infty (\log \nu)^{-1} \nu^{-s},$$

where $s = 1 + it$, on every fixed segment $|t| < T$ of the line $\sigma = 1$. The steps in his proof are reversible and he concludes that the uniform convergence of the series (2.2) on every fixed segment $|t| < T$ of the line $\sigma = 1$ is equivalent to the prime number theorem,

$$p_n \sim n \log n.$$

3. The Fourier Integral

The Fourier integral is the repeated integral

$$(3.1) \quad \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t) \cos \mu (t - x) \, dt \, d\mu.$$

The (C, r) sum of this integral is defined by

$$(3.2) \quad I^{(r)} = \lim_{n \to \infty} \frac{1}{\pi} \int_0^n \int_{-\infty}^\infty (1 - \frac{\mu}{n})^r f(t) \cos \mu(t - x) \, dt \, d\mu.$$

Carslaw [9, 1926] considered first the special case

$$f(x) = 1, \quad (0 < x < \ell); \; f(x) = -1, (-\ell < x < 0)$$

and then the general case where $f(x)$ satisfies $f(x) = \frac{1}{2}\{f(x+) + f(x-)\}$, to prove that, in each case, the Gibbs phenomenon for (3.2) is essentially indicated by the Gibbs phenomenon for the integral

$$(3.3) \quad \int_0^x (1 - \frac{\mu}{x})^r \frac{\sin \mu}{\mu} \, d\mu.$$

Comparing (3.3) with the corresponding result for the Gibbs phenomenon of the (C, r) means of ordinary Fourier series, it follows that the (C, r) means of the Fourier integral display the Gibbs phenomenon for $r < r_o$, where r_o is Cramér's constant.

4. The Fourier-Bessel Series*

The Fourier-Bessel series expansion of a function $f(t)$ is defined by

$$(4.1) \qquad f(t) \sim \sum A_n J_\nu(j_n t)$$

where

$$(4.2) \qquad A_n = \frac{2}{J^2_{\nu+1}(j_n)} \int_o^1 \mu J_\nu(j_n\mu)\, f(\mu) d\mu,$$

and j_n is the n^{th} positive zero of $J_\nu(\mu)$.

Young** proved that, except at the end points $t = 0$ and $t = 1$, the convergence properties of the Fourier-Bessel series and of the ordinary Fourier series are identical, so that if a function $f(t)$ has a discontinuity at a point internal to this interval, its Gibbs ratio is given by

$$(4.3) \qquad \frac{2}{\pi} \int_o^\pi \frac{\sin \mu}{\mu}\, d\mu.$$

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* In sections 4 and 10 of this chapter, summation is always with respect to n.

** Young, W. H., On series of Bessel functions. Proc. London Math. Soc., (2), 18 (1920).

Cooke [13, 1928] proved that (4.3) is also the Gibbs ratio at $t = 1$. To obtain the Gibbs ratio at $t = 0$, he considered the modified series

$$(4.4)\qquad x^{\alpha} \sum \frac{2\, J_{\nu}(j_n t)}{J^2_{\nu+1}(j_n)} \int_0^1 J_{\nu}(j_n\mu)\mu^{1-\alpha} f(\mu)\, d\mu,$$

which reduces to (4.1) for $\alpha = 0$, and to ordinary Fourier series for $\alpha = \nu = 1/2$. He proved the following theorem:

The modified Fourier-Bessel series (4.4) has a Gibbs ratio at $t = 0$ equal to

$$(4.5)\qquad 2^{1-\alpha} \frac{\Gamma(1 - 1/2(\alpha-\nu))}{\Gamma(1/2(\alpha+\nu))} \int_0^{j_1} J_{\nu}(\mu)\, \mu^{\alpha-1}\, d\mu$$

where j_1 is the smallest positive zero of $J_{\nu}(\mu)$, provided that $-\frac{1}{2} < \alpha \leq 1$, $\alpha - \nu < 2$, $\alpha + \nu > 0$, $f(0+) \neq 0$. If $\alpha + \nu = 0$, $\alpha > -\frac{1}{2}$, there is no Gibbs phenomenon at $x = 0$. As a special case, if $\nu = \frac{1}{2}$, we have the result that the Fourier series

$$(4.6)\qquad \frac{2t^{\alpha-\frac{1}{2}}}{\pi} \sum \sin n\pi t \int_0^1 \mu^{\frac{1}{2}-\alpha} f(\mu) \sin n\pi\mu\, d\mu, \qquad f(0+) \neq 0$$

has a Gibbs ratio at $t = 0$ equal to

$$(4.7)\qquad \frac{2^{3/2-\alpha}}{\pi^{1/2}} \frac{\Gamma(5/4 - \alpha/2)}{\Gamma(1/4 + \alpha/2)} \int_0^{\pi} \mu^{\alpha-3/2} \sin\mu\, d\mu$$

provided that $-\frac{1}{2} < \alpha < 3/2$; and if $\alpha = \frac{1}{2}$, this reduced to

$$(4.8)\qquad 2/\pi \int_0^{\pi} \frac{\sin\mu}{\mu}\, d\mu.$$

5. The Hankel Integral

The Hankel Integral is defined by

$$(5.1) \quad \int_0^\infty \int_0^\infty J_\nu(x\mu)\ J_\nu(t\mu)\ t\mu\ f(t)\ d\mu\ dt$$

and in the modified form by

$$(5.2) \quad x^\alpha \int_0^\infty \int_0^\infty J_\nu(x\mu)\ J_\nu(t\mu)\ t^{1-\alpha}\mu\ f(t)\ d\mu dt$$

The modified form (5.2) reduces to (5.1) for $\alpha = 0$. The Hankel integral reduces to the Fourier integral for $x > 0$ on the substitution of the asymptotic form of the Bessel functions.

Cooke [13, 1928] pointed out that for $\nu = 1/2$, (5.1) reduces to the Fourier integral of $x^{1/2} f(x)$, hence at a point of ordinary discontinuity other than the origin, the Gibbs ratio is given by (4.8). To obtain the ratio at the origin, he considered the modified form (5.2) and proved the following theorem:

> The modified Hankel integral (5.2) has a Gibbs ratio equal to (4.5) provided that $-\frac{1}{2} < \alpha \leq 1$, $\alpha - \nu < 2$, $\alpha + \nu > 0$, $f(0+) \neq 0$. If $\alpha + \nu = 0$, $\alpha > -\frac{1}{2}$, there is no Gibbs phenomenon at $x = 0$. As a special case, if $\nu = \frac{1}{2}$, we have that the Fourier integral
>
> $$(5.3) \quad \frac{2x^{\alpha-\frac{1}{2}}}{\pi} \int_0^\infty \int_0^\infty t^{\frac{1}{2}-\alpha} \sin x\mu \sin t\mu\ f(t)\ dt\ d\mu, \qquad f(0+) \neq 0,$$
>
> has a Gibbs ratio given by (4.7) provided that $-\frac{1}{2} < \alpha < 3/2$; and if $\alpha = \frac{1}{2}$, this further reduces to (4.8).

6. The Hermite-Fourier Series

Let $\{x^n e^{-x^2/2}\}$ be a sequence of functions defined on the entire real line. The Gram-Schmidt orthogonalization process applied to these functions yields the set $\{H_\nu(c)\}$ of normalized Hermite functions. The Hermite-Fourier series of a function $f(x)$ is then given by

$$(6.1) \qquad f(x) \sim \sum c_\nu H_\nu(x)$$

where

$$(6.2) \qquad c_\nu = (2^\nu \nu!\, \pi^{1/2})^{-1} \int_{\infty}^{\infty} e^{-\mu^2} f(\mu)\, H_\nu(\mu)\, d\mu.$$

That the Gibbs phenomenon for the arithmetic mean of the partial sums of the series (6.1) does not vanish was first proved by Kogbetliantz [54, 1932]. He also proved that it does not vanish when Cesáro's method of order r, $r > 0$, is applied, obtaining in this case the Gibbs ratio

$$(6.3) \qquad 1 + 2^{1+r} \frac{\Gamma(1+r)}{\pi} \int_0^\infty \frac{\sin \mu \, d\mu}{\{\mu + \pi(1+r/2)\}^{r+1}} > 1.$$

Jacob [51, 1937] modified Kogbetliantz's work to obtain the following more precise results for these series:

> Let the functions $f(x)$ and $e^{x^2/2}\,|f(x)/x|$ be integrable, respectively, for $|x| \le a_1$ and $a_2 \le |x| \le \infty$, the positive members a_1 and a_2 being as large as we like, but fixed;

and let (6.1) be the development of $f(x)$ in a series of Hermite polynomials. If $f(x)$ has a jump at $x = a$, and is continuous in some neighborhood of this point, then in summing this series by the Cesáro (C, r) method, that is, in forming the sum

$$(6.4) \qquad f_n(x) = \sum_{o}^{n} \left(1 - \frac{\nu}{n}\right)^r c_\nu H_\nu(x),$$

the Gibbs ratio is given by

$$(6.5) \qquad 2^{r+1/2} \frac{\Gamma(r+1)}{\pi^{1/2}} \int_o^{j_1} \frac{J_{r+1/2}(\mu)}{\mu^{r+1/2}} d\mu > 1, \qquad r > 0.$$

where j_1 is the first positive zero of the integrand.

Thus it is seen that the Gibbs phenomenon occurs for the (C, r) means for all r greater than zero. However, on the application of the weight function $g_\nu = \nu^{-\frac{1}{2}}$ to the means, he proved the following theorem:

Under the hypothesis of the above theorem, in summing the series (6.1) by Cesáro's method, with weight function $g_\nu = \nu^{-\frac{1}{2}}$; that is, in forming the sum

$$(6.6) \qquad f_n(x) = \sum_{o}^{n} \left(1 - \left(\frac{\nu}{n}\right)^{\frac{1}{2}}\right)^r c_\nu H_\nu(x),$$

the Gibbs ratio is given by

$$(6.7) \qquad 2 \frac{\Gamma(r+1)}{\pi} \int_o^{j_1} \frac{C_{r+1}(\mu)}{\mu^{r+1}} d\mu$$

where j_1 is again the first positive zero of the integrand and $C_{r+1}(\mu)$ is Young's function.

Now $C_{r+1}(\mu)$ is positive for all $r \geq 1$; hence $j_1 = \infty$, so that for $r \geq 1$, the Gibbs phenomenon does not present itself. For $0 < r < 1$, there exists an r_o such that there is no Gibbs phenomenon for $r \geq r_o$, but there is a Gibbs phenomenon for $r < r_o$, r_o being Cramér's constant. Jacob does not state explicitly that r_o is Cramér's constant, but from the sense of his concluding remarks, he seems to imply that it is.

7. The Singular Integral $\int f(t)\nu\, J_\nu(\nu t)\, dt$

Let $f(t)$ be defined in $(0, A)$, $A > 1$, and suppose that $t^\lambda f(t)$ is Lebesgue integrable over $(0, A)$ for some constant λ, $f(1-)$ exists and $f(t)$ is of bounded variation in $[1, A]$. Under these hypotheses, Lorch and Szego [76, 1955] proved that

$$(7.1) \qquad \lim_{\nu \to \infty} \int_0^A f(\mu)\nu\, J_\nu(\nu\mu)\, d\mu = 1/3\, f(1-) + 2/3\, f(1+)$$

If the condition that $f(t)$ is bounded variation in $[1, A]$ is replaced by the condition that $f(1+)$ exist, and if $p(\nu) > 0$ is such that $\tau = 1/3 \{ \lim_{\nu \to \infty} \nu\{2p(\nu)\}^{3/2}\}$ exists, then they proved that

$$(7.2) \qquad \lim_{\nu \to \infty} \int_0^{1+p(\nu)} f(\mu)\nu\, J_\nu(\nu\mu)\, d\mu = 1/3\, f(1-) + 2/3\, f(1+)\, G_\tau$$

where

$$(7.3) \qquad G_\tau = 1/2 \int_0^\tau \{J_{1/3}(\mu) + J_{-1/3}(\mu)\}\, d\mu, \qquad 0 \leq \tau < \infty .$$

G_τ attains its maximum value of $1.4115282\ldots$ at $\tau = 2.3834466\ldots$, the first positive zero of the integrand. Its minimum value is zero for $\tau = 0$. For $\tau > 2.3834466$, G_τ oscillates about $G_\tau = 1$, attaining minimum and maximum values at alternate zeroes of the integrand. The integral (7.2) is thus seen to exhibit a Gibbs phenomenon, attaining a maximum value of $1/3\ f(1-) + (0.9410188\ldots)f(1+)$, and a minimum value of $1/3\ f(1-)$.

The authors proved [77, 1962] that the Lebesgue constants, L_n, satisfy the inequality

$$(7.4) \qquad L_n > kn^{1/2}$$

for some positive constant k, and announced the result that

$$(7.5) \qquad L_n = (2/\pi)^{3/2} \left\{\int_1^A (\mu^2 - 1)\, d\mu\right\} n^{1/2} + O(1), \qquad n \to \infty .$$

For the (C, r) means, the n^{th} Lebesgue constant is given by

$$(7.6) \qquad L_n(r, 0) = rn^{-r} \int_0^A \left| \int_0^n (n - \mu)^{r-1} \mu\, J_\mu(\mu t)\, d\mu \right| dt.$$

To evaluate this expression, the authors replaced $J_\mu(t)$ in the kernel by the circular function

$$(7.7) \qquad C_\mu(t) = J_\mu(t) \cos \alpha - Y_\mu(t) \sin \alpha, \qquad 0 \leq \alpha < \pi,$$

where $Y_\mu(t)$ is the Bessel function of the second kind:

$$(7.8) \qquad Y_\mu(t) = \{J_\mu(t) \cos \pi\mu - J_{-\mu}(t)\}/\sin \pi\mu$$

if μ is not a whole number; and

$$(7.9)\quad Y_\mu(t) = \lim_{\gamma \to \mu} \{J_\gamma(t)\cos\pi\gamma - J_{-\gamma}(t)\}/\sin\pi\gamma$$
$$= \frac{1}{\pi} \lim_{\gamma \to 0} \frac{(-1)^\nu J_{\gamma-\mu}(t) - J_{\mu-\gamma}(t)}{\gamma}$$

if μ is a whole number. Denoting the corresponding Lebesgue constants for the (C, r) means by $L_n^*(r, \alpha)$, they proved that

$$(7.10)\quad L_n^*(r, \alpha) = (2/\pi)^{3/2}\Gamma(r+1)n^{1/2-r}\int_a^b \sec\mu(\tan\mu)^{1/2}(\tan\mu-\mu)^{-r}d\mu+O(1)$$

where $a = n^{-1/3}$ and $b = \operatorname{arcsec} A$. Considering this integral separately for $r = 0$, $r = 1/2$, and $0 \neq r \neq 1/2$, they proved that

$$(7.11)\quad L_n^*(r, \alpha) = (2/\pi)^{3/2}\,\Gamma(r+1)n^{1/2-r}\int_0^b \sec\mu(\tan\mu)^{1/2}(\tan\mu - \mu)^{-r}d\mu + O(1), \qquad 0 \leq r \leq 1/2$$

$$(7.12)\qquad L_n^*(1/2, \alpha) = (2/3)^{1/2}.\ \pi^{-1}\log n + O(1)$$

$$(7.13)\qquad L_n^*(r, \alpha) = O(1) \quad \text{for} \quad r > 1/2.$$

This result is seen to be independent of α, so that setting $\alpha = 0$, we obtain $C_\mu(t) = J_\mu(t)$, and (7.11), (7.12), (7.13) then give the Lebesgue constants $L_n(r, 0)$ for the (C, r) means of the singular integral. It is seen that these constants are unbounded unless $r > 1/2$. This implies that the integral is (C, r) summable to $1/3\ f(1-) + 2/3\ f(1+)$ everywhere if $r > 1/2$, but not necessarily if $r \leq 1/2$.

8. The Jacobi-Fourier Series

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, α, β real, greater than -1, may be defined in terms of the generating function

$$(8.1) \qquad 2^{\alpha+\beta} \varphi^{-1}(1-\omega-\varphi)^{-\alpha} (1+\omega+\varphi)^{-\beta} = \sum P_\nu^{(\alpha,\beta)}(x)\, \omega^\nu$$

where $\varphi = (1 - 2x\omega + \omega^2)^{1/2}$. They may also be defined by the relation

$$(8.2) \qquad (1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, n!} \frac{d^n}{dx^n}(1-x)^{n+\alpha}(1+x)^{n+\beta} .$$

Each polynomial $P_n^{(\alpha,\beta)}(x)$ is of n^{th} order in x, and these polynomials form an orthonormal set on the interval $[-1, 1]$. Hence any function $f(x)$, Lebesgue integrable on $[-1, 1]$, may be expressed in a Jacobi-Fourier series

$$(8.3) \qquad f(x) \sim \sum c_\nu P_\nu^{(\alpha,\beta)}(x),$$

where the constants c_ν are given by

$$(8.4) \qquad c_\nu = \int_{-1}^{1} f(\mu)\, P_\nu^{(\alpha,\beta)}(\mu)\, d\mu .$$

The Lebesgue constants, $L_n(\alpha, \beta)$, for the Jacobi series (8.3) were first derived by Rau [89, 1929] as generalizations of Fejer's earlier result [20, 1910], for ordinary Fourier series, and of Gronwall's result [29, 1913] for the Legendre series. Rau obtained the form

$$(8.5) \qquad L_n(\alpha, \beta) = \int_{-1}^{1} (1 - x)^\alpha(1 + x)^\beta |s_n^{(\alpha,\beta)}(x)|\, dx$$

or, by setting $x = \cos\theta$,

$$(8.6) \qquad L_n(\alpha,\beta) = \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \int_0^\pi (\sin\theta/2)^{2\alpha+1}(\cos\theta/2)^{2\beta+1} \cdot |P_n^{(\alpha+1,\beta)}(\cos\theta)|\, d\theta \quad .$$

He then proved the following:

$$(8.7) \qquad L_n(\alpha,\beta) = An^{\alpha+1/2} + o(n^{\alpha+1/2}), \qquad n\to\infty,\ \alpha > -1/2$$

where
$$A = \frac{\Gamma(\alpha/2 + 1/4)\,\Gamma(\beta/2 + 3/4)}{\Gamma(\alpha+1)\,\Gamma(1/2(\alpha+\beta)+1)}$$

$$(8.8) \qquad L_n(\alpha,\beta) = O(\log n), \qquad n\to\infty,\ \alpha = -1/2$$

$$(8.9) \qquad L_n(\alpha,\beta) = O(1), \qquad n\to\infty,\ -1 < \alpha < -1/2 \ .$$

For $\alpha = \beta = 0$, the Jacobi polynomials reduce to the Legendre polynomials, and (8.7) reduces to Gronwall's result [29, 1913] for the Legendre series,

$$(8.10) \qquad L_n(0,0) = 2\sqrt{2/\pi}\ n^{1/2} + o(n^{1/2}), \qquad n\to\infty \ .$$

For $\alpha = \beta = -1/2$, the Jacobi polynomials reduce to the Tchebychef polynomials, and the series are then equivalent to ordinary Fourier series. (8.7) then reduces to Fejér's result for ordinary convergence:

$$(8.11) \qquad L_n(-1/2,-1/2) = \frac{4}{\pi^2}\log n + o(\log n), \qquad n\to\infty \ .$$

Rau also mentioned two results obtained and later published by Szegö [110, 1933]:

$$(8.12) \qquad L_n(-1/2,\beta) = \frac{4}{\pi^2}\log n + o(\log n), \qquad n\to\infty$$

$$(8.13)\qquad L_n(\alpha,\ \beta) = \frac{2^{-\alpha}}{\Gamma(\alpha+1)} \int_0^{\infty} \mu^{\alpha}\ |J_{\alpha+1}(\mu)|\ d\mu + o(1); \qquad -1 < \alpha < -1/2.$$

Lorch [71, 1959] sharpened Rau's result (8.7) somewhat in the following theorem:

$$(8.14)\qquad L_n(\alpha,\ \beta) = An^{\alpha+1/2} + O(n^{\alpha-1/2}) + O(n^{\alpha-\beta-1}) + O(1), \qquad n \to \infty,$$

for $\alpha > -1/2$, provided that $\alpha \neq 1/2$, $\beta \neq -1/2$. If $\alpha = 1/2$, then $O(n^{\alpha-1/2})$ must be replaced by $O(\log n)$, and if $\beta = -1/2$, then $O(n^{\alpha-\beta-1})$ must be replaced by $O(n^{\alpha-1/2} \log n)$. The constant A is the same as in (8.7).

He was able to generalize this to the following:

$$(8.15)\qquad L_n(\alpha,\ \beta,\ \mu,\ \lambda)$$

$$= \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \int_0^{\pi} (\sin\ \theta/2)^{\mu}(\cos\ \theta/2)^{\lambda}|P_n^{(\alpha+1,\beta)}(\cos\ \theta)|d\theta$$

$$= A_{\mu\lambda}n^{\alpha+1/2} + O(n^{\alpha+\beta-\lambda}) + O(n^{2\alpha-\mu+1}) + O(n^{\alpha-1/2}) + O(1)$$

for $\mu > \alpha + 1/2$, $\lambda > \beta - 1/2$, provided that $\mu - \alpha \neq 3/2$ and $\lambda - \beta \neq 3/2$. If $\mu - \alpha = 3/2$, then $O(n^{2\alpha-\mu+1}) + O(n^{\alpha-1/2})$ must be replaced by $O(n^{\alpha-1/2} \log n)$. If $\lambda - \beta = 1/2$, then $O(n^{\alpha+\beta-\lambda}) + O(n^{\alpha-1/2})$ must be replaced by $O(n^{\alpha-1/2} \log n)$. Here,

$$A_{\mu\lambda} = \frac{2}{\pi^{3/2}} \quad \frac{\Gamma(\mu/2 - \alpha/2 - 1/4)\ \Gamma(\lambda/2 - \beta/2 + 1/4)}{\Gamma(\alpha + 1)\ \ \Gamma(\mu/2 + \lambda/2 - \alpha/2 - \beta/2)} \quad .$$

His first theorem is then a specialization of the last one, obtained by taking $\mu = 2\alpha + 1$, $\lambda = 2\beta + 1$.

Subsequently, Lorch [72, 1959] sharpened his results for the restricted range $-1/2 < \alpha < 1/2$, $\alpha - \beta < 1$, and for the case $\alpha = -1/2$. For the first case, he proved that

$$(8.16) \quad L_n(\alpha, \beta) = An^{\alpha+1/2} + B\alpha + O(m^{\alpha-1/2}) + O(n^{\alpha-\beta-1}), \qquad n \to \infty,$$

where A is the same constant as before, and $B\alpha$ is a constant term which depends only on α. For the second case, he obtained

$$(8.17) \quad L_n(-1/2, \beta) = \frac{4}{\pi^2} \log n + C_\beta + O(n^{-1} \log n) + O(n^{-\beta-3/2}), \; n \to \infty,$$

where now C_β is a constant which depends only on β.

9. The Legendre-Fourier Series

$$\text{Set} \quad f_n(x) = (1 - x^2)^n \; ; \quad \varphi_n(x) = (-1)^n(2^n n!)^{-1} \frac{d^n}{dx^n} f_n(x).$$

The sequence $\{\varphi_n\}$ defines a sequence of what are called Legendre polynomials, orthonormal on $[-1, 1]$, which are a specialization of the Jacobi polynomials, corresponding to the case $\alpha = \beta = 0$. A Lebesgue-integrable function on $[-1, 1]$ can be expanded in a Legendre-Fourier series

$$(9.1) \qquad f(x) \sim \sum c_\nu \varphi_\nu(x),$$

where the constants c_ν are given by

$$(9.2) \qquad c_\nu = \int_{-1}^{1} f(\mu)\, \varphi_\nu(\mu)\, d\mu.$$

In the course of his study of the Laplace series, Gronwall [29, 1913] proved that the Lebesgue constants, $L_n(0, 0)$, for the Legendre series are given by

$$(9.3) \qquad L_n(0, 0) = 2\sqrt{\frac{2}{\pi}}\, n^{1/2} + o(n^{1/2}), \qquad n \to \infty .$$

This result is also a specialization of Rau's result (8.10) for the Jacobi series, corresponding to the case $\alpha = \beta = 0$.

10. The Schlömilch Series

The theory of Schlömilch series * is concerned with an examination of the conditions under which it is possible to expand an arbitrary function $f(x)$ in a series of the form **

$$(10.1) \qquad \sum A_n(x) = \sum \frac{a_n J_\nu(nx) + b_n H_\nu(nx)}{(1/2\, nx)^\nu}$$

where $H_\nu(t)$ is Struve's function and $J_\nu(t)$ is the Bessel function. For $\nu > -1/2$,

$$(10.2) \qquad H_\nu(t) = \frac{t^\nu}{2^{\nu-1}\,\Gamma(1/2)\,\Gamma(\nu+1/2)} \int_0^{\pi/2} \sin(t \sin\theta)\cos^{2\nu}\theta\, d\theta$$

$$(10.3) \qquad J_\nu(t) = \frac{t^\nu}{2^{\nu-1}\,\Gamma(1/2)\,\Gamma(\nu+1/2)} \int_0^{\pi/2} \cos(t \sin\theta)\cos^{2\nu}\theta\, d\theta$$

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* Watson, G. N., The Theory of Bessel Functions, 1922, Chapter XIX

** see footnote, page 91.

The coefficients a_n, b_n are given by

$$(10.4) \qquad a_n = \int_{-\pi}^{\pi} \int_0^{\pi/2} \frac{\sec^{2\nu+1} t}{\Gamma(1/2-\nu)\Gamma(1/2)} \frac{d}{dt} \left\{ \sin^{2\nu} t \{ f(\mu \sin t - f(0) \} \right\} \cos n\mu \, dt \, d\mu$$

$$(10.5) \qquad b_n = \int_{-\pi}^{\pi} \int_0^{\pi/2} \frac{\sec^{2\nu+1} t}{\Gamma(1/2-\nu)\Gamma(1/2)} \frac{d}{dt} \left\{ \sin^{2\nu} t \{ f(\mu \sin t - f(0) \} \right\} \sin n\mu \, dt \, d\mu.$$

Wilton [123, 1926] considered the special case of a series of the type

$$(10.6) \qquad f(x) \sim \sum n^{-1/2} \varphi(n) J_{\nu+1/2}(xn),$$

where the summation is over n. He proved that

1. If $0 < a < \pi$, $\delta > 0$ and as $n \to \infty$,

$$(10.7) \qquad \varphi(n) = c \cos(an - 1/2\nu\pi) + O(n^{-\delta}),$$

where ν is an arbitrary complex number; and if

$$0 < \epsilon \leq x \leq 2\pi - a' < 2\pi - a,$$

then the function $f(x)$ defined by the series (10.6) has a single ordinary discontinuity at $x = a$, with saltus $c(\pi/2a)^{1/2}$; and the series exhibits, in the same measure as the Fourier series, the Gibbs phenomenon as $x \to a$.

2. If $-1/2 < \lambda < \nu+1$, c is real, $\delta > 0$, and ϵ is the greater of the two members δ and $\delta + 1/2 - \lambda$; and if

$$\varphi(n) = an^{-\lambda} + O(n^{-\lambda-\epsilon}) \quad \text{as} \quad n \to \infty, \tag{10.8}$$

and $0 \leq x \leq 2\pi - \kappa < 2\pi$ and $f(x)$ is defined by the series

$$f(x) \sim \sum x^{1-\lambda} \varphi(n) \; J_\nu\{x(n + c)\}, \tag{10.9}$$

then $f(0) = 0$, $f(0+) = a\ell$, where

$$\ell = 2^{-\lambda} \, \Gamma(\nu/2 - \lambda/2 + 1/2)/\Gamma(\nu/2 + \lambda/2 + 1/2) \; , \tag{10.10}$$

and the series exhibits the Gibbs phenomenon as $x \to 0$. There is no other discontinuity in the interval $(0, 2\pi - \kappa)$.

For the (C, r) means of the series (10.9), Wilton proved the following theorem:

If $r > 0$, the (C, r) sum of the series (10.9) does or does not exhibit the Gibbs phenomenon as $x \to 0$ according as there is or is not a number $\xi > 0$ such that

$$\int_0^1 (1 - t)^r \; \xi^{1-\lambda} \; J_\nu(\xi t) t^{-\lambda} dt > \ell \; , \tag{10.11}$$

where ℓ is defined by (10.10).

Cooke [14, 1928] proved that if $0 < \epsilon < \pi$, and $e_1 = [\epsilon, \pi)$, $e_2 = (-\pi, -\epsilon]$, then the Schlömilch series (10.1) behaves with respect to convergence, divergence, oscillation, uniform or otherwise, precisely like

the Fourier series of $f(x)$ at every point of e_1 and e_2. Thus the Gibbs ratio at points of simple discontinuity in e_1 and e_2 is equal to

$$(10.12) \qquad \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} \, dt.$$

At $x = 0$ he considered the modified Schlömilch series $x \sum A_n(x)$ for which he proved that the Gibbs ratio is given by

$$(10.13) \qquad 2^{\nu} \frac{\Gamma(\nu + 1 2)}{\Gamma(1/2)} \int_0^{j_1} t^{-\nu} J_{\nu}(t) \, dt,$$

where j_1 is the least positive zero of $J_{\nu}(t)$.

Cooke [15, 1929] considered the series

$$(10.14) \qquad \sum a_n \frac{J_{\nu}(nx)}{(1/2\, nx)^{\nu}}$$

where

$$(10.15) \qquad a_n = \frac{2n}{\Gamma(1/2)\Gamma(1/2-\nu)} \int_0^{\pi} \int_0^{\pi/2} \mu \, f(\mu \sin t) \sin^{2\nu+1} t \sec^{2\nu} t \sin n\mu \, dt \, d\mu$$

and proved the following theorem:

The modified Schlomilch series

$$(10.16) \qquad x^{\alpha+\nu} \sum a_n \frac{J_{\nu}(nx)}{(1/2\, nx)^{\nu}}, \qquad |\nu| < 1/2,$$

has a Gibbs ratio at $x = 0$ equal to

$$(10.17) \qquad 2^{1-\alpha} \Gamma(1-\alpha/2+\nu/2)/\Gamma(\alpha/2+\nu/2) \int_0^{j_1} t^{\alpha-1} J_{\nu}(t) \, dt,$$

provided that $\alpha + \nu > 0$, $-1/2 < \alpha \leq 1$, $f(0+) \neq 0$.

Cooke [16, 1930] also obtained the Gibbs ratio for the Schlomilch series of Struve functions

$$(10.18) \qquad \sum b_n \frac{H_\nu(nx)}{(1/2\ nx)^\nu}, \qquad |\nu| < 1/2$$

where

$$(10.19) \qquad b_n = \frac{-2n}{\Gamma(1/2)\Gamma(1/2-\nu)} \int_0^\pi \int_0^{\pi/2} \mu\, f(\mu \sin t) \sin^{2\nu+1} t \sec^{2\nu} t \cos n\mu\, dt\, d\mu .$$

For points of ordinary discontinuity other than the origin, the Gibbs ratio is given by (10.12). At $x = 0$, he proved that it is given by

$$(10.20) \qquad \frac{2^{\nu+1}}{\pi} \Gamma(\nu+1) \max \int_0^\tau t^{-\nu-1} H_\nu(t)\, dt, \qquad \tau > 0.$$

He tried to show that (10.20) has a maximum at $\tau = j_1$, the least positive zero of $H_\nu(t)$, but states that the work became overpowering, and he proved, instead, that there is no Gibbs phenomenon for the series (10.18) for $\nu \geq -0.10$, and that there is one for $-1/2 \leq \nu \leq -0.20$.

11. <u>The Tchebychef-Fourier Series</u>

The Tchebychef polynomials $\{T_n(\theta)\}$ may be defined by

$$(11.1) \qquad T_0(\theta) = \pi^{-1/2}, \quad T_n(\theta) = (2/\pi)^{1/2} \cos(n \cos^{-1}\theta), \qquad n = 1,2,\ldots.$$

They may be derived by applying the Gram-Schmidt orthogonalization process to the sequence $\{x^n\}$, $n = 0,1,2, \ldots$, and setting $x = \cos\theta$. They are orthonormal on [-1, 1].

Let

$$(11.2) \qquad f(\theta) \sim \sum c_\nu T_\nu(\theta)$$

and let

$$(11.3) \qquad s_n(\theta) = \sum_{o}^{n} c_\nu T_\nu(\theta)$$

be the n^{th} partial sum of the Tchebichef-Fourier series of $f(\theta)$, continuous in $[-1, 1]$. The coefficients c_ν are given by

$$(11.4) \qquad c_\nu = \int_{-1}^{1} f(\mu)\, T_\nu(\mu)(1-\mu^2)^{-1/2}\, d\mu \,.$$

The delayed arithmetic means of the partial sums (11.3) are given by

$$(11.5) \qquad \sigma_{n,p}(\theta) = (p+1)^{-1} \sum_{n-p}^{n} s_\nu(\theta),$$

and the corresponding Lebesgue constants by

$$(11.6) \qquad L_{n,p}(\theta) = \sup |\sigma_{n,p}(\theta)|, \qquad |f| \leq 1.$$

Abramov [1, 1954] proved that

$$(11.7) \quad L_{n,p}(\theta) = \frac{4}{\pi^2} \left\{\log \frac{1+ny}{1+(p+1)y} + |\cos ny| \log \frac{y+(n+1)^{-1}}{y+n^{-1}} \right\} + O(1)$$

where $y = \cos^{-1}\theta$.

12. The Titchmarsh Integral

The Titchmarsh integral* is the integral given by

$$(12.1) \qquad \int_0^\infty Y_\nu(x\mu)\mu \, H_\nu(t\mu)t \, f(t) \, dt \, d\mu$$

and in the modified form by

$$(12.2) \qquad x^\alpha \int_0^\infty Y_\nu(x\mu)\mu \, H_\nu(t\mu)t^{1-\alpha} \, f(t) \, dt.$$

(12.2) reduces to (12.1) for $\alpha = 0$, and to the Fourier sine series for $\alpha = 1/2$, $\nu = -1/2$. Pertaining to the Gibbs phenomenon, Cooke [12, 1928] obtained the following result:

The modified Titchmarsh integral (1.2) has a Gibbs ratio at $x = 0$ equal to

$$(12.3) \qquad 2^{1-\alpha} \frac{\Gamma(1-\alpha/2+\nu/2)}{\Gamma(\alpha/2+\nu/2)} \tan(1 - \alpha/2 + \nu/2)\pi \int_0^{j_p} Y_\nu(t)t^{\alpha-1} \, dt,$$

where $p = 1$ if $-1 < \nu < -1/2$ and $p = 2$ if $-1/2 \leq \nu < 1/2$, provided that $f(0+) \neq 0$, and when $-1 < \nu < -1/2$, we have $1/2 < \alpha < 3/2$, $\alpha + \nu > 0$, $1 < \alpha - \nu < 2$; when $-1/2 \leq \nu < 1/2$, we have $\alpha + \nu > 0$, $1 < \alpha - \nu < 2$, $\alpha < 3/2 - \{(1 - 4\nu^2)/(2\pi^2)\}$. The upper limit of integration j_p is the p^{th} positive zero of $Y_\nu(t)$.

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* Titchmarsh, E. C., Extensions of Fourier's Integral formula to formulae involving Bessel functions. Proc. London Math. Soc. (2) 23 (1924).

At points other than the origin, the Gibbs ratio is given by

$$(12.4) \qquad \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

In a subsequent paper [16, 1930], Cooke states that the Gibbs ratio for the associated integral

$$(12.5) \qquad \int_0^{\infty} \int_0^{\infty} H_\nu(x\mu)\mu \; Y_\nu(t\mu)t \; f(t)\, dt\, d\mu$$

is given by

$$(12.6) \qquad \frac{2^{\nu+1}}{\pi} \Gamma(\nu + 1) \;\max \int_0^{\tau} t^{-\nu-1} H_\nu(t)\, dt, \qquad \tau > 0 .$$

He also proved the result that there is no Gibbs phenomenon for the integral (12.5) for $\nu \geq -0.10$, and that there is one for $-1/2 \leq \nu \leq 0.20$.

13. The Walsh-Fourier Series

The Radamacher functions are defined by

$$(13.1) \qquad \varphi_o(x) = 1 \quad (o \leq x < 1/2) \qquad \varphi_o(x) = -1 \quad (1/2 \leq x < 1)$$

$$\varphi_o(x + 1) = \varphi_o(x) \qquad \varphi_n(x) = \varphi_o(2^n x), \qquad n = 1, 2, \ldots$$

These functions form an orthonormal system on [0, 1]. However, they are not complete. Their completion was constructed by Walsh [116, 1923] by the introduction of what are now known as the Walsh functions:

$$(13.2) \qquad \psi_o(x) = \varphi_o(x); \quad \psi_n(x) = \varphi_{n_1}(x)\varphi_{n_2}(x) \;\ldots\; \varphi_{n_\nu}(x),$$

where $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\nu}$, the integers n_i being uniquely determined by the relation $n_{i+1} < n_i$. Every periodic function $f(x)$ which is Lebesgue integrable on [0, 1] can be expanded in a Walsh-Fourier series

$$(13.3) \qquad f(x) \sim \sum c_\nu \psi_\nu(x) ,$$

where the coefficients are given by

$$(13.4) \qquad c_\nu = \int_0^1 \psi_\nu(\mu)\, f(\mu)\, d\mu.$$

The partial sums $s_n(x)$ and the kernel $D_n(x, \mu)$ are given, respectively, by

$$(13.5) \qquad s_n(x) = \int_0^1 f(\mu) \sum_0^{n-1} \psi_\nu(\mu)\, \psi_\nu(x)\, d\mu$$

$$(13.6) \qquad D_n(x, \mu) = \sum_0^{n-1} \psi_\nu(\mu)\, \psi_\nu(x).$$

Walsh proved that the Lebesgue constants, L_n, for the Walsh-Fourier series are unbounded. The full details of these constants were worked out by Fine [21, 1949], whose results are summarized in the following theorem:

The Lebesgue constants, L_n, for the Walsh system satisfy

$$(13.7) \qquad L_n = \nu - \sum_{1 \leq p < r \leq \nu} 2^{n_r - n_p}$$

where $n = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_\nu}$, $\qquad n_1 > n_2 > \ldots > n_\nu \geq 0.$

$$(13.8)\qquad L_{2n} = L_n \ ; \qquad L_{2n+1} = 1/2\{1 + L_n + L_{n+1}\}$$

$$(13.9)\qquad L_n = O(\log n)$$

$$(13.10)\qquad n^{-1}\sum_1^n L_\nu = \frac{\log n}{4 \log 2} + O(1)$$

$$(13.11)\qquad \limsup \ \{L_n - (4/9 + 1/3 \log_2 3n)\} = 0, \qquad n \to \infty$$

$$(13.12)\qquad \sum L_\nu x^\nu = 1/2 \frac{x}{(1-x)^2} \sum \frac{1}{2^\nu} \frac{1-x^{2^\nu}}{1+x^{2^\nu}}, \qquad |x|<1 .$$

Some results on the Lebesgue constants for this system were also announced by Šneider [94, 1950], but his work is not available at the time that this is being written.

BIBLIOGRAPHY

[1] Abramov, L. M., On the Asymptotic Behaviour of the Lebesgue Functions of certain methods of summation of Tchebychef functions. Dokl. Akad. Nauk SSSR (NS), 98 (1954), 173-176.

[2] Barlaz, J., On some triangular summation methods. Amer. Jour. of Math., 69 (1947), 139-152.

[3] Bôcher, M., Theory of Fourier series. Annals of Math., (1906), 123-132.

[4] Bôcher, M., On the Gibbs phenomenon. Jour. für Math., Berlin, 144 (1914).

[5] Campbell, R., Sur les summations de Cesáro d'ordre entier des série de Weber. C. R. Acad. Sci. Paris, 233 (1951).

[6] Carlitz, L., Note on Lebesgue constants. Proc. Amer. Math. Soc., 12 (1961), 932-935.

[7] Carslaw, H. S., A trigonometrical sum and the Gibbs phenomenon in Fourier series. Amer. Jour. of Math., 39 (1917).

[8] Carslaw, H. S., Fourier's Series and Integrals. MacMillan and Co., Ltd., London (1921), 264-282.

[9] Carslaw, H. S., Gibbs phenomenon in the sum (C, r) for $r > 0$, of Fourier's Integral. Jour. London Math. Soc., 1 (1926), 201-204.

[10] Cheng, M., The Gibbs phenomenon and Bochner's summation method I. Duke Math. Jour., 17 (1950), 83-90.

[11] Cheng, M., The Gibbs phenomenon and Bochner's summation method II. Duke Math, Jour., 17 (1950), 477-490.

[12] Cooke, R. G., A case of Gibbs phenomenon. Jour. London Math. Soc., 3 (1928), 92-98.

[13] Cooke, R. G., Gibbs phenomenon in Fourier-Bessel series and integrals. Proc. London Math. Soc., 27 (1928), 171-192.

[14] Cooke, R. G., On the theory of Schlömilch series. Proc. London Math. Soc., 28 (1928), 207-241.

[15] Cooke, R. C., Gibbs phenomenon in Schlömilch series. Jour. London Math. Soc., 4 (1929), 18-21.

[16] Cooke, R. G., Disappearing Gibbs Phenomenon. Proc. London Math. Soc. 30 (1930), 144-164.

[17] Cowling, V. F., Summability and Analytic Continuation. Proc. Amer. Math. Soc., 1 (1950), 536-542.

[18] Cramér, H., Etudes sur le summation des séries de Fourier. Arkiv För Mat., Astron. och Fysik, 13 (1919), 1-21.

[19] Farvard, J., Sur quelques problems de convergence dans theorie de l'approximation des functions d'une variable reelle. Celebrazione Archimedes de Sec. XX (Syracuse, 1962), V. II, Edizione Oderize, Gubbio, 1962.

[20] Fejér, L., Lebesguesche Konstanten and Divergente Fourierreihen. Jour. für die Reine und Angew. Math. 138 (1910), 22-53.

[21] Fine, N. J., On the Walsh functions. Trans. Amer. Math. Soc., 65 (1949), 372-414.

[22] Forbes, R. L., Gibbs phenomenon. M.Sc. Thesis, University of Alberta, (1962).

[23] Ganea, T., Gibbs phenomenon in the theory of Fourier series. Gaz. Mat. Fiz., Ser. A, 8 (1956), 10-18.

[24] Gibbs, J. W., Fourier Series. Nature 59 (1899), 606.

[25] Goes, G., BK-Raüme und Matrixtransformation für Fourierkoefficienten. Math. Zeitschr, 70 (1958-1959).

[26] Govil, N. K., A note on Gibbs phenomenon for the (λ) means of Fourier series. Ind. Jour. Math., 5 (1963), 37-40.

[27] Gronwall, T. H., Über die Gibbssche Erscheinung und die trigonometrischen summen $\sum n^{-1} \sin nx$. Math. Annalen, 72 (1912), 228-243.

[28] Gronwall, T. H., Über die Lebesgueschen Konstanten bei den Fourierschen Reihen. Math. Ann., 72 (1912), 244-261.

[29] Gronwall, T. H., Über die Laplaceschen Reihe. Math. Annalen, 74 (1913), 213-270.

[30] Gronwall, T. H., Zur Gibbsschen Erscheinung. Annals of Math., 31 (1930), 233-240.

[31] Hardy, G. H., Note on Lebesgue constants in the theory of Fourier series. Jour. London Math. Soc., 17 (1942), 4-13.

[32] Hardy, G. H., Divergent series. Oxford, 1949.

[33] Hardy, G. H. and Rogosinski, W. W., Notes on Fourier series II, On the Gibbs phenomenon. Jour. London Math. Soc., 18 (1943), 83-87.

[34] Hardy, G.H., and Rogosinski, W.W., Fourier Series. Cambridge Tracts in Math. and Phys., No. 38, Cambridge, 1944.

[35] Harsiladze, F.I., Gibbs phenomenon in the summation of Fourier series by methods of Bernstein-Rogosinski. Dokl. Acad. Nauk, SSSR(NS), 101 (1955), 425-428.

[36] Hsiang, F.C., On the Gibbs phenomenon for Harmonic Means. Proc. Amer. Math. Soc. 11 (1960), 284-290.

[37] Hsiang, F.C., Gibbs phenomenon for fractional integration. Portug. Math., 21 (1962), 31-36.

[38] Hylten-Cavalius, C., Geometrical methods applied to trigonometric sums. Proc. Roy. Physiog. Soc. Lund., 21 (1950).

[39] Ishiguro, K., Fourier Series XI, Gibbs phenomenon. Kodai Math. Sem. Rep. 8 (1956), 181-188.

[40] Ishiguro, K., Correction to the paper Fourier Series XI, Gibbs phenomenon. Kodai Math. Sem. Rep. 9 (1957), 191-192.

[41] Ishiguro, K., Fourier Series XV, Gibbs phenomenon. Proc. Jap. Acad., 33 (1957), 119-123.

[42] Ishiguro, K., On the Gibbs phenomenon. Jour. Fac. Science, Hokkaido Univ., Ser. 1, 15 (1960), 145-171.

[43] Ishiguro, K., The Lebesgue constants for (γ,r) summation of Fourier series. Proc. Jap. Acad., 36 (1960), 470-474.

[44] Ishiguro, K., Zur Gibbsschen Erscheinung für das Kreisverfahren. Math. Zeitschr., 76 (1961), 288-294.

[45] Ishiguro, K., Über das S_α Verfahren bei Fourier-Reihen. Math. Zeitschr., 80 (1962), 4-11.

[46] Ishiguro, K., On the Quasi-Hausdorff means whose weight function has jumps. Proc. Jap. Acad., 40 (1964), 59-64.

[47] Ishiguro, K., On the Lebesgue constants for Quasi-Hausdorff methods of summability. Proc. Jap. Acad., 40 (1964), 188-195.

[48] Ishiguro, K., and Kuttner, B., On the Gibbs phenomenon for Quasi-Hausdorff means. Proc. Jap. Acad., 39 (1963), 731-735.

[49] Izumi, S. and Satô, M., Fourier Series X. Rogosinski's Lemma. Kodai Math. Sem., Rep. 8 (1956), 164-180.

[50] Izumi, S. and Satô, M., Fourier series XVI. The Gibbs phenomenon of partial sums and Cesáro means of Fourier series, I and II. Proc. Jap. Acad., 33 (1957), 284-292.

[51] Jacob, M.M., Sur le phénomène de Gibbs dans le development de séries de polynomes d'Hermite. C.R. Acad. Sci. Paris, 204 (1937), 1540-1543.

[52] Jakimovski, A., A generalization of the Lototsky method of summability. Mich. Math. Jour., 6 (1959), 277-290.

[53] Knopp, K., Theory and Application of Infinite Series. Hafner Publ. Co., 2nd Edition (Trans. of the 4th German Edition, 1947).

[54] Kogbetliantz, M.E., Annales de l'Ecole Normale. Ser. 3, 49 (1932), 137-221.

[55] Kuttner, B., On the Riesz means of Fourier series II. Jour. London Math. Soc., 19 (1944), 77-84.

[56] Kuttner, B., On the Gibbs phenomenon for Riesz means. Jour. London Math. Soc., 19 (1944), 153-161.

[57] Kuttner, B., Note on the Gibbs phenomenon. Jour. London Math. Soc., 20 (1945), 136-139.

[58] Kuttner, B., Further note on the Gibbs phenomenon. Jour. London Math. Soc., 22 (1947), 295-298.

[59] Lebesgue, H., Lecons sur les Séries Trigonométriques. Paris, 1906.

[60] Lee, C., On the Gibbs phenomenon for Riemann summation (R, 1) of Fourier series. Acta Math. Sinica. 6 (1956), 418-425.

[61] Lee, C., On the Gibbs phenomenon for Riemann summation (R, k) of generalized Fourier series. Acta Math. Sinica, 9 (1959), 28-36.

[62] Livingston, A.E., Some Hausdorff means which exhibit the Gibbs phenomenon. Pac. Jour. Math., 3 (1953), 407-415.

[63] Livingston, A.E., The Lebesgue constants for Euler (E, p) summation of Fourier series. Duke Math. Jour., 21 (1954), 309-314.

[64] Livingston, A.E., Gibbs phenomenon for some Hausdorff means. Address to the Amer. Math. Soc., Feb., 1963.

[65] Lorch, L., Lebesgue constants for Borel summability. Duke Math. Jour., 11 (1944), 459-467.

[66] Lorch, L., On Fejér's calculation of Lebesgue constants. Bul. Calcutta Math. Soc., 37 (1945), 5-8.

[67] Lorch, L., The Lebesgue constants for (E, 1) summation of Fourier series. Duke Math. Jour., 19 (1952), 45-50.

[68] Lorch, L., Asymptotic expressions for some integrals which include certain Lebesgue and Fejér constants. Duke Math. Jour., 20 (1953), 89-103.

[69] Lorch, L., The principal term in the asymptotic expansion of the Lebesgue constants. Amer. Math. Monthly, 61 (1954), 245-249.

[70] Lorch, L., The Gibbs phenomenon for Borel means. Proc. Amer. Math. Soc., 8 (1957), 81-84.

[71] Lorch, L., The Lebesgue constants for Jacobi series I. Proc. Amer. Math. Soc., 10 (1959), 756-761.

[72] Lorch, L., The Lebesgue constants for Jacobi series II. Amer. Jour. of Math., 81 (1959), 875-888.

[73] Lorch, L. and Newman, D. J., Lebesgue constants for Regular Hausdorff methods. Can. Jour. of Math., 13 (1961), 283-298.

[74] Lorch, L. and Newman, D. J., On the $[F, d_n]$ summation of Fourier series. Comm. Pure and App. Math., 15 (1962), 109-118.

[75] Lorch, L. and Newman, D. J., The Lebesgue constants for (γ, r) summation of Fourier series. Can. Math. Bull., 6 (1963), 179-182.

[76] Lorch, L. and Szego, P., A singular integral whose kernel involves a Bessel function I. Duke Math. Jour., 22 (1955), 407-418.

[77] Lorch, L. and Szego, P., A singular integral whose kernel involves a Bessel function II. Acta. Math., Acad. Scien. Hung., 13 (1962), 203-217.

[78] Matsumoto, K., Lebesgue's constant of (R, λ, κ) summation. Proc. Jap. Acad., 32 (1956), 658-661.

[79] Matušů, J., On a type of integral with the so-called Gibbs phenomenon. Cesk. Akad. Vid., Aplik. Mat., 6 (1961), 245-262.

[80] Miracle, C. L., The Gibbs phenomenon for Taylor means and for $[F, d_n]$ means. Can. Jour. Math., 12 (1960), 660-673.

[81] Moore, C. N., On the application of Borel's method to the summation of Fourier's series. Proc. Nat. Acad. Sci., 11 (1925), 284-287.

[82] Natanson, I. P., Gibbs phenomenon in the general theory of Singular Integrals. Izves. Vys. Uched. Zaved., SSSR, 40 (1964).

[83] Newman, D. J., Gibbs phenomenon for Hausdorff means. Pac. Jour. Math., 12 (1962), 367-370.

[84] Pavlov'skiǐ, M. M., Riemann sums for integrals of the moduli of certain trigonometric polynomials. Dop. Akad. Nauk URSR (1962), 722-728.

[85] Prachar, K. and Schmetterer, L., Über die Euler'sche Summierung Fourier'scher Reihen . Anzeiger des Öster. Akad. Wissen., Math. Klasse, 85 (1948), 33-39.

[86] Prasad, B. N. and Siddiqi, J. A., On the Gibbs phenomenon for Nörlund means. Ind. Jour. Math., 1 (1958), 21-27.

[87] Rabson, G., Summability of Fourier series on the quaternions of norm one. Trans. Amer. Math. Soc., 75 (1953), 287-303.

[88] Radamacher, H., Einige Satz über Reihen von algemeinen orthogonalfunktionen. Math. Ann., 87 (1922), 112-138.

[89] Rau, H., Über die Lebesgueschen Konstanten der Reihenentwicklungen nach Jacobischen Polynomen. Jour. für die Reine und Angew. Math., 161 (1929), 427-450.

[90] Runge, C., Theorie und Praxis der Reihen. G. J. Goschen'sche Verlagshandlung, (1904).

[91] Šerbina, A. D., On a generalization of the method of Fejér for the summation of Fourier series. Dokl. Akad. Nauk SSSR (NS), 60 (1948).

[92] Shrivastava, K. C., The generalized jump of a function and Gibbs phenomenon. Ann. Mat. Pura Appl., 52 (1960), 331-347.

[93] Sledd, W. T., The Gibbs phenomenon and Lebesgue constants for regular Sonnenschein matrices. Can. Jour. Math., 14 (1962), 723-728.

[94] Sneĭder, A. A., On the convergence of subsequences of Walsh functions. Dokl. Akad. Nauk SSSR (NS), 70 (1950).

[95] Sokolov, I. G., The remainder term in the Fourier series of differentiable functions. L'vov Gos. Univ. Uc. Zap. 29, Ser. Meh. Mat., 6 (1954).

[96] Stečkin, S. B., On de la Vallee-Poussin sums. Dokl. Akad. Nauk SSSR, 80 (1951).

[97] Stečkin, S. B., Some remarks on trigonometric polynomials. Uspehi Mat. Nauk (NS), 10 (1955), 159-166.

[98] Stein, E. M., Localization and summability of multiple Fourier series. Acta Math., 100 (1958), 93-147.

[99] Stein, E. N., On certain sums arising in multiple Fourier series. Annals of Math., 73 (1961), 87-109.

[100] Szász, O., On the partial sums of Fourier series at points of discontinuity. Trans. Amer. Math. Soc., 53 (1943), 440-453.

[101] Szász, O., On some trigonometrical summability methods and Gibbs phenomenon. Trans. Amer. Math. Soc., 54 (1943), 483-497.

[102] Szász, O., Generalized jump of a function and Gibbs phenomenon. Duke Math. Jour., 11 (1944), 823-833.

[103] Szász, O., On the Gibbs phenomenon for Euler means. Acta Scient. Math., Szeged, 12 (1950), 107-111.

[104] Szász, O., Gibbs phenomenon for Hausdorff means. Trans. Amer. Math. Soc., 69 (1950), 440-456.

[105] Szász, O., On a summation method of O. Perron. Math. Zeitschr., 52 (1950), 631-636.

[106] Szász, O., Tauberian theorems for summability (R_1). Amer. Jour. Math., 73 (1951), 779-791.

[107] Szász, O., On the Gibbs phenomenon for a class of linear transforms. Akad. Serbe. Sci., Publ. Inst. Math., 4 (1952), 135-144.

[108] Szegö, G., Über die Lebesgueschen Konstanten bei den Fourierschen Reihen. Math. Zeitschr., 9 (1921), 163-166.

[109] Szegö, G., Über die Enwicklung einer willkürlichen Function nach den Polynomen einer Orthogonal systems. Math. Zeitschr., 12 (1922), 61-94.

[110] Szegö, G., Asymptotische Entwicklungen der Jacobischen Polynome. Schr. Konigsberg. Gel. Ges. Naturwiss. Kl., 10 (1933), 35-102.

[111] Szegö, G., Orthogonal Polynomials. Amer. Math. Soc. Colloquium Publications. Vol. 23 (1939).

[112] de Sz. Nagy, B., Sur une classe générale de procédés de sommation pour les séries die Fourier. Hung. Acta Math., 1 (1948).

[113] Tandori, K., Sur les constants de Lebesgue des systemes de fonctions orthogonales et normees. C. R. Akad. Sci., Paris, 244 (1957), 1128-1130.

[114] Timan, A. F., On the Lebesgue constants for certain methods of summability. Dokl. Akad. Nauk SSSR (NS), 61 (1948).

[115] Walmsley, C., Gibbs phenomenon for Cesáro and Hölder summation of generalized Fourier series. Jour. London Math. Soc., 28 (1953), 148-156.

[116] Walsh, J. L., A closed set of normal, orthogonal functions. Amer. Jour. Math., 55 (1923), 5-24.

[117] Watson, G. N., The constants of Landau and Lebesgue. Quart. Jour. of Math., 1 (1930), 310-318.

[118] Weber, H., Über die Gibbssche Erscheinung bei bestimmten integralen. Math. Annalen, 73 (1913), 286-288.

[119] Weyl, H., Die Gibbssche Erscheinung in der theorie der Kugelfunctionen. Palermo, Rend. Circ. Math., 29 (1910).

[120] Weyl, H., Über die Gibbsche Erscheinung und verwandke Konvergenz phänomene. Palermo, Rend. Circ. Math., 30 (1910).

[121] Widder, D. V., The Laplace Transform. Princeton Univ. Press, (1946).

[122] Wilbraham, H., On a certain periodic function. Cambridge and Dublin Math. Jour., 3 (1848), 198-201.

[123] Wilton, J. R., The Gibbs phenomenon in series of Schlömilch type. Messenger of Math., 56 (1926).

[124] Wintner, A., Gibbs phenomenon and the prime number theorem. Amer. Jour. Math., 67 (1945), 167-172.

[125] Zalcwasser, Z., Sur le phénomène de Gibbs des la théorie des séries de Fourier des fonctions continues. Fund. Math., 12 (1928), 126-151.

[126] Zeller, K., Theorie der Limitierungsverfahren. Springer-Verlag, Berlin, (1958).

[127] Zygmund, A., Trigonometric Series. Cambridge Univ. Press, (1959), Vol. I.